THE DYNAMIC OF BELIEF IN THE TRANSFERABLE BELIEF MODEL AND SPECIALIZATION-GENERALIZATION MATRICES.

F. Klawonn and Ph. Smets

Department of Computer Science
Technical University of Braunschweig
Braunschweig-Germany

IRIDIA
Université Libre de Bruxelles
Brussels-Belgium

Abstract: The fundamental updating process in the transferable belief model is related to the concept of specialization and can be described by a specialization matirx. The degree of belief in the truth of a proposition is a degree of *justified support*. The Principle of Minimal Commitment implies that one should never give more support to the truth of a proposition than justified. We show that Dempster's rule of conditioning corresponds essentially to the least committed specialization, and that Dempster's rule of combination results essentially from commutativity requirements. The concept of generalization, dual to the concept of specialization, is described.

Keywords. Transferable belief model, belief function, Dempster-Shafer theory, dynamic of belief, specialization, Dempster's rule of conditioning, Dempster's rule of combination.

Words Count: 4469 words.

1. Introduction.

The dynamic of belief in the Dempster-Shafer theory is achieved by the application of Dempster's rule of conditioning and Dempster's rule of combination. Their appropriateness has been questioned (Levi 1983, Kyburg 1987, Pearl 1990, Voorbracht 1991). But most criticisms result from the fact that the Dempster-Shafer theory is understood as a special form of upper and lower probability theory.

In response to these criticisms, one of us (Smets 1988, 1991b, Smets and Kennes 1990) developped the transferable belief model (TBM), a model to quantify Your[1] belief. The TBM is build without using any concept of probability, hence the criticisms against the use of Dempster's rules do not apply (Smets, 1991a). But there is still the question of explaining why they should be used. This paper presents such a justification.

We present the needed material about the TBM in section 2. In section 3, we introduce the **Principle of Minimal Commitment**. It consists in never giving more support to the

[1]'You' is the agent that entertains the beliefs considered in this presentation.

elements of the domain of our beliefs than justified. When necessary, it permits us to select the least informative belief function in a set of equally justified belief functions.

In section 4, we study the dynamic components of the TBM, and in particular the expansion process, i.e. the addition of a belief without retracting any old beliefs (Gardenfors 1988). In the TBM, this expansion process is modelled by **specialization** matrices, i.e. operators that reallocate the basic belief masses that quantify our beliefs. The justification of both Dempster's rules is presented in section 5. It proceeds as follows.

When conditioning on an event A, the updated beliefs should give a null plausibility to $\overline{A}$ as $\overline{A}$ becomes impossible. The specialization matrix that satisfies that requirement and that induces the minimal commitment updating leads to an updated belief identical to the one obtained by the application of Dempster's rules of conditioning.

It seems natural to require that combinations of pieces of evidence commute among them and with any conditioning. Dempster's rule of combination is the unique soloution satisfying these commutativity requirements.

These results provide justifications for the use of Dempster's rules when beliefs must be expanded by new pieces of evidence.

2. The transferable belief model.

The transferable belief model covers the same domain of application as the subjective probability model except that beliefs are quantified by belief functions, not by probability functions. Let Ω be a finite set of elements, called the frame of discernment. At most one of the elements of Ω corresponds to the actual state of affair, but we do not know which one. We can only assess the degrees of belief that this particular element belongs to the various subsets of Ω.

In the transferable belief model, a unitary amount of belief is postulated, and it is distributed over the subsets of Ω. The part of belief m(A) given to A, called a **basic belief mass** (bbm), quantifies the part of our belief that supports A and does not support any strict subset of A due to lack of information. The bbm m(A) for $A \subseteq \Omega$ are non negative and

$$\sum_{A \subseteq \Omega} m(A) = 1.$$

The function $m: 2^{\Omega} \rightarrow [0,1]$ is called a **basic belief assignment** (bba). The **degree of belief** bel(A) allocated to a subset A of Ω is the sum of the parts of beliefs m(X) given to subsets X of A that are not subsets of $\overline{A}$:

$$bel(A) = \sum_{\emptyset \neq X \subseteq A} m(X)$$

The **degree of plausibility** pl(A) quantifies the maximum amount of belief that might support A:

$$pl(A) = bel(\Omega) - bel(\overline{A}) = \sum_{X \cap A \neq \emptyset} m(X)$$

The **communality function** $q:2^{\Omega} \rightarrow [0,1]$ is defined as

$$q(A) = \sum_{A \subseteq B} m(B)$$

It does not have any practical interpretation but is most useful to represent Dempster's rule of combination (see below).

A **vacuous belief function** is a belief function such that $bel(A)=0\ \forall A \neq \Omega$, and $bel(\Omega) = 1$. It quantifies our belief in a state of total ignorance. The corresponding bba (where $m(\Omega)=1$) and plausibility function (where $pl(A) = 1\ \forall A \subseteq \Omega$), are called the **vacuous bba and plausibility function**.

Basic belief assignments, belief functions, plausibilty functions and commonality functions are in one to one correspondance (Shafer 1976). The same indexing will be used to show which functions are linked together through that correspondence.

One should note that no normalization is introduced, $m(\emptyset)$ may be positive, and $bel(\Omega)$ and $pl(\Omega)$ may be less than one. The conclusions still apply when normalization is introduced.

3. The Principle of Minimal Commitment.

Let $\Omega = \{a, b, c\}$. Suppose You know only that My[1] belief function over Ω is such that $bel_{Me}(\{a\}) = .3$ and $bel_{Me}(\{b,c\}) = .5$, and You do not know the value I give to bel_{Me} for the other subsets of Ω. Suppose You have no other information on Ω and You are ready to adopt My belief as Yours. How to build Your belief given these partial constraints. Many belief functions can satisfy them. If you adopt the principle that subsets of Ω should not get more support than justified, then Your belief on Ω will be such that $m_{You}(\{a\})=.3$, $m_{You}(\{b,c\})=.5$ and $m_{You}(\{a,b,c\})=.2$. Among all belief functions compatible with the constraints given by known vlaues of bel_{Me}, bel_{You} is the one that gives the smallest degree of belief to every subsets of Ω. The principle evokes here is called the **Principle of Minimal Commitment**. It is really at the core of the TBM, where degrees of belief are degrees of 'justified' supports.

With un-normalized belief functions, the principle definition is essentially based on the plausibilty function. Suppose two plausibilty functions pl_1 and pl_2 such

$$pl_1(A) \leq p_2l(A) \qquad \forall A \subseteq \Omega.$$

We say that pl_2 is not more committed than pl_1 (and less committed if there is at least one strict inequality). The same qualification is extended to their related bba and belief functions. Among all belief functions on Ω, the least committed belief function is the vacuous belief function (i.e. its bbm $m(\Omega)=1$).

[1] 'I' (or 'Me') is an agent different from 'You'.

When expressed with belief functions, the principle becomes:

$$\mathrm{bel}_1(A) + m_1(\emptyset) \geq \mathrm{bel}_2(A) + m_2(\emptyset) \qquad \forall A\subseteq\Omega.$$

The concept of 'least commitment' permit the construction of a partial order $\sqsubseteq$ on the set of belief functions (Yager 1986, Dubois and Prade 1986).

We write:

$$pl_1 \sqsubseteq pl_2$$

to denote that pl_1 is equal to pl_2 or more committed than pl_2. By analogy the following notations $m_1 \sqsubseteq m_2$ and $\mathrm{bel}_1 \sqsubseteq \mathrm{bel}_2$ have the same meaning.

The **Principle of Minimal Commitment** consists in selecting the least committed belief function in a set of equally justified belief functions. The principle formalizes the idea that one should never give more support than justified to any subset of Ω. It satisfies a form of scepticism, of uncommitment, of conservatism in the allocation of our belief. In its spirit, it is not far from what the probabilists try to achieve with the maximum entropy principle (see Dubois and Prade 1987, Hsia, 1991, Smets 1991c)

4. The dynamic of the transferable belief model.

In order to build the operators for conditioning and combinations, we formalize the concept of evidential corpus (see Levi 1980 for a similar corpus). We define EC as the **evidential corpus,** i.e. the set of all Your evidence at a given time. Two EC are said to be equivalent (relative to Ω) if the bba they induce on Ω are equal to each other, even after they have been updated by the addition of any new piece of evidence to EC. We say that an evidence E is compatible with EC if $EC\cup E \not\equiv \perp$ where $\perp$ is the contradiction. We say that an evidence E is vacuous (relative to Ω) if EC and $EC\cup E$ are equivalent (relative to Ω) for every EC. A tautology is a typical example of a vacuous evidence.

Among others, change in Your beliefs can be due to expansion, i.e. the addition to EC of an evidence that is compatible with EC. We analyse in detail the expansion process within the TBM where it is represented by two rules: the **Dempster rule of conditioning** and the **Dempster rule of combination**. We explain first the nature of the two rules before introducing the concept of a specialization matrix that will represent the general operator corresponding to the expansion.

Consider first the expansion of Your EC by the fact that all elements of $\overline{A}$ are impossible. It corresponds to the conditioning process on A. As You learn that the elements in $\overline{A}$ are now impossible, the bbm m(X) that was supporting X supports now $X\cap A$, so m(X) is transfered to $X\cap A$. This transfer of bbm results in the unnormalized **Dempster rule of conditioning**. The updated bba m_A and belief function bel_A are:

$$m_A(B) = \sum_{\emptyset \neq Y \subseteq \overline{A}} m(B \cup Y) \qquad \text{for } B \subseteq A$$
$$\qquad 0 \qquad \text{otherwise}$$

and $\quad bel_A(B) = bel(B \cup \overline{A}) - bel(\overline{A}) \qquad \text{for } B \subseteq \Omega$

Now let EC be Your evidential corpus and let E be a non-vacuous evidence (relative to Ω). We say that E is distinct from EC if there is no non-vacuous evidence F such that E⊨F and EC⊨F.

Let the evidence E be distinct from the evidential corpus EC. Let m_0 and m_1 be the bba on Ω induced by EC and E, respectively. The problem is to build the bba m_{01} induced by EC∪E. Shafer (1976) proposed to compute m_{01} from m_0 and m_1 by applying **Dempster's rule of combination**. The ⊕ symbol is also used to denote this combination. So

$$m_{01} = m_0 \oplus m_1$$

and $\quad bel_{01} = bel_0 \oplus bel_1$

where bel_{01} is the belief function induced by the bba m_{01} with

m_{01}(A) = \i\su(Y∩Z=∅ Y,Z\S\UP1(⊆) Ā,, m_0(A∪Y)m_1(A∪Z))

∀A⊆Ω

In particular one has:

$$q_{01}(A) = q_0(A)\, q_1(A) \qquad \forall A \subseteq \Omega$$

Both rules have already received some justifications presented without assuming any underlying probabilistic model. For the conditioning rule, Nguyen and Smets (1991) show that Bel(A|B) is the minimal commitment solution for the belief of the conditional object A|B (Goodman, Nguyen et al., 1991, Goodman, Gupta et al.1991). For the combination rule, Smets (1990), Klawonn and Schwecke (1990), Dubois and Prade (1986) provide sets of algebraic requirements that justify it. Another justification based on the concepts of commutative specialization is presented here.

5. Specializations.

The concept of specialization is at the core of the transferable belief model interpretation of the belief allocation by belief functions. Let m_0 be a bba induced on Ω by Your initial EC. The impact of adding a new evidence E to EC induces a change in Your beliefs characterized by a redistribution of the initial m_0 such that the bbm m(A) given to the subset A of Ω is distributed among the subsets of A. Let s(A,B)∈ [0, 1] be the proportion of m(A) that flows into B⊆A when the new evidence E is added to EC. In order to conserve the whole mass m(A) after this transfer, the s(A,B) must satisfy:

$$\sum_{B \subseteq A} s(A,B) = 1 \qquad \forall A \subseteq \Omega$$

As masses can flow only to subsets, then s(A,B) = 0 ∀B⊄A. The matrix S of the coefficient s(A,B) for A,B⊆Ω is called a **specialization matrix** on Ω (see Yager 1986, Dubois and Prade 1986, Kruse and Schwecke 1990, Moral 1985, Delgado and Moral 1987).

After the expansion of EC by E, the initial bba m_0 is transformed into the new bba m such that:

$$m(A) = \sum_{X \subseteq \Omega} s(X,A)\ m_0(X)$$

The bba m is called a specialization of m_0. We write $m = m_0 \cdot S$ to represent the specialization of m_0 into m by S.

Whenever a bba m is a specialization of a bba m_0, then m is at least as committed as m_0 (Yager 1986). So $m_0 \cdot S \sqsubseteq m_0$ for any bba m_0 and any specialization matrix S (all defined on Ω).

It is easy to show that the effects of both Dempster's rules can be obtained by specialization matrices.

Let $m:2^{\Omega} \rightarrow [0,1]$ be a bba and Σ be the set of specialization matrices on Ω.

1) To obtain **Dempster's rule of conditioning**, let $C \subseteq \Omega$ and let S_C be the specialization matrix such that

$$s_C(A,B) = \begin{cases} 1 & \text{if } B = A \cap C \\ 0 & \text{otherwise} \end{cases}$$

Let m_C be the bba obtained after conditioning the bba m on C by Dempster's rule of conditioning. Then $m_C = m \cdot S_C$. We call S_C the C-conditionating specialization matrix. We define $\Sigma_{cond} \subseteq \Sigma$ as the set of C-conditionating specializations where $C \subseteq \Omega$.

2) To obtain **Dempster's rule of combination**, let m be a bba on Ω and let S_m be a specialization matrix with coefficients

$$s_m(A,B) = m_A(B) \qquad \forall A,B \subseteq \Omega$$

where m_A is the bba obtained after conditioning the bba m on A by Dempster's rule of conditioning. The coefficients $s_m(A,B)$ satisfy:

$$s_m(\Omega,B) = m(B) \qquad \forall B \subseteq \Omega$$

and

$$s_m(A,B) = \begin{cases} \sum_{X \subseteq \overline{A}} s_m(\Omega, B \cup X) & \forall B \subseteq A \\ 0 & \text{otherwise} \end{cases}$$

Suppose two bba m and m'. One can prove that $m' \oplus m = m' \cdot S_m$.

S_m is called the **Dempsterian specialization matrix associated with m** as it updates any m' on Ω into $m \oplus m'$. We define $\Sigma_D \subseteq \Sigma$ as the set of the Dempsterian specialization matrices.

In particular, if m is such that m(C) = 1 (the bba that corresponds to a conditioning on C), then $S_C = S_m$, so $\Sigma_{cond} \subseteq \Sigma_D$.

We consider that the expansion procedure by the specialization is one of the fundamental ideas for the dynamic part of the transferable belief model. Accepting that every expansion is defined by a specialization matrix, we will show that :

1. When conditioning on $A \subseteq \Omega$, $S_A \in \Sigma_{cond}$ is the specialization matrix that induces the least committed updated plausibility on Ω such that the updated plausibility given to $\overline{A}$ is 0. The requirement $pl(\overline{A})=0$ after expansion translates the fact that all elements of Ω in $\overline{A}$ are impossible.

2. Σ_D is the largest family of specialization matrices that commute and that includes Σ_{cond}. The commutativity translates the idea that the expansion by two pieces of evidence should lead to the same result whatever the order with which they are considered. It is classically required for any expansion operator (Gardenfors 1988).

6. Dempster's rules in the view of specializations.

We first show that $S_A \in \Sigma_{cond}$ is the specialization matrix that induces the least committed updated plausibility on Ω such that the updated plausibility given to $\overline{A}$ is 0.

Theorem 1: **Dempster's rule of conditioning.**
Let $C \subseteq \Omega$ and let $\Sigma^* = \{S \in \Sigma \mid \forall m\text{: if } m'=m \cdot S \text{ then } pl'(\overline{C})=0\}$, where pl' denotes the plausibility function induced by $m'=m\cdot S$. Let S_C be the C-conditionating specialization matrix with

$$s_C(A,B)=1 \qquad \text{if } B=A\cap C \text{ and}$$
$$s_C(A,B)=0 \qquad \text{otherwise.}$$

Then $S_C \in \Sigma^*$ and the bba $m\cdot S_C$ is the least committed bba among the bba $m\cdot S$ for $S \in \Sigma^*$:

$$m\cdot S_C \sqsubseteq m\cdot S \qquad \forall S \in \Sigma^*.$$

Proof. Let m be a bba over Ω. Let $S \in \Sigma^*$. Let $m'=m\cdot S$. One must have

$$0 = pl'(\overline{C}) = \sum_{B\cap\overline{C}\neq\emptyset} m'(B) = \sum_{B\cap\overline{C}\neq\emptyset} \sum_{B\subseteq A} m(A)\cdot s(A,B)$$

It must be true for all bba m on Ω. Therefore $s(A,B)=0$ for all B satisfying $B \not\subseteq C$ and for all $A \subseteq \Omega$. Let S be such a specialization matrix.

$$pl'(D) = \sum_{B\cap D\neq\emptyset} m'(B) = \sum_{B\cap D\neq\emptyset} \sum_{B\subseteq A} m(A)\cdot s(A,B)$$
$$= \sum_{B\subseteq C,\ B\cap D\neq\emptyset} \sum_{B\subseteq A} m(A)\, s(A,B) = \sum_{A\cap D\cap C\neq\emptyset} m(A) \sum_{B\subseteq A\cap C,\ B\cap D\neq\emptyset} s(A,B)$$
$$\leq \sum_{A\cap D\cap C\neq\emptyset} m(A)$$

For any m, let pl_C be the plausibility function induced by $m_C=m\cdot S_C$.

$$pl_C(D) = \sum_{B\cap D\neq Ø} m_C(B) = \sum_{B\cap D\neq Ø} \sum_{B\subseteq A} m(A)\, s_C(A,B) = \sum_{A\cap C\cap D\neq Ø} m(A)$$

So whatever the bbm m on Ω, $\forall D\subseteq\Omega$, $pl_C(D)\geq pl'(D)$. So pl_C is the least committed plausibility function among those induced by $m\cdot S$ for $S\in\Sigma^*$. QED

Since Dempster's rule of conditioning is idempotent, so are the conditionating specialization matrices.

Lemma 1: The specializations matrices $S\in\Sigma_{cond}$ are idempotent.

We prove that Σ_D is the largest set of specialization matrices that commute among themselves and that includes Σ_{cond}.

Theorem 2: **Dempster's rule of combination.**
Let $S_1\in\Sigma$ such that for all $C\subseteq\Omega$: $S_1S_C=S_CS_1$ with $S_C\in\Sigma_{cond}$. Then $S_1\in\Sigma_D$ holds.

Proof.Let $S_{1,C} = S_1S_C$ with elements $s_{1,C}(A,B)$:

$$s_{1,C}(A,B) = \sum_{D\subseteq\Omega} s_1(A,D)\cdot s_C(D,B) = \sum_{X\subseteq\overline{C}\cap A} s_1(A,B\cup X)\cdot s_C(B\cup X,B)$$

$$= \sum_{X\subseteq\overline{C}\cap A} s_1(A,B\cup X) \quad \text{if } B\subseteq C$$

$$= 0 \quad \text{otherwise}$$

Let $S_{C,1} = S_CS_1$ with elements $s_{C,1}(A,B)$:

$$s_{C,1}(A,B) = \sum_{D\subseteq\Omega} s_C(A,D)\cdot s_1(D,B) = s_1(A\cap C,B)$$

Thus for all $A,B,C\subseteq\Omega$, one must have:

$$s_1(A\cap C,B) = \sum_{X\subseteq\overline{C}\cap A} s_1(A,B\cup X) \quad \text{if } B\subseteq C$$

$$= 0 \quad \text{otherwise}$$

If we set $A=\Omega$, we obtain for all $B,C\subseteq\Omega$,

$$s_1(C,B) = \sum_{X\subseteq\overline{C}} s_1(\Omega,B\cup X) \quad \text{if } B\subseteq C$$

$$= 0 \quad \text{otherwise}$$

which implies that S_1 is Dempsterian. QED

Since the Dempster rule of combination is associative and commutative, the Dempsterian specialization matrices commute.

Theorem 3: If $S_{m_1}, S_{m_2} \in \Sigma_D$, then $S_{m_1} \cdot S_{m_2} = S_{m_2} \cdot S_{m_1}$.

Proof: $m_0 \cdot S_{m_1} \cdot S_{m_2} = (m_0 \oplus m_1) \cdot S_{m_2} = (m_0 \oplus m_1) \oplus m_2 = m_0 \oplus m_1 \oplus m_2 = m_0 \oplus m_2 \oplus m_1 = (m_0 \oplus m_2) \cdot S_{m_1} = m_0 \cdot S_{m_2} \cdot S_{m_1}$ QED

Note: Let m be a bba on Ω, with q its related commonality function. Let S_m be the Dempsterian specialization matrix generated by m. Let T be the matrix that transforms a bba into a commonality function, where the elements $t_{A,B}$ of T are:

$$t_{A,B} = \begin{cases} 1 & \text{if } B \subseteq A \subseteq \Omega \\ 0 & \text{otherwise} \end{cases}$$

It can be shown that 1°) the communalities q(A): $A \subseteq \Omega$, are the diagonal elements and the eigenvalues of S_m and 2°) the lines of T^{-1} are the eigenvectors of S_m[1]. One has the representation: $S_m = T \, \Lambda \, T^{-1}$

where Λ is a diagonal matrix with elements $\lambda_{A,A} = q(A)$, $A \subseteq \Omega$. Among others, this property may be useful when $m(\Omega) = 0$, in which case the theory of generalized inverses described in matrix calculus can be helpful.

7. An alternate characterization of Dempster's rule of combination.

To understand this theorem, consider that My EC induces the bba m_0 on Ω. I want to build a specialization matrix S ready to combine the impact of any new evidence distinct from EC, i.e. to specialize any new bba m on Ω. Let Σ_0 be the set of the specialization matrices S such that the application of S on any m on Ω would be at least as committed as m_0. One of the elements of Σ_0 is S_{m_0}, the Dempsterian specialization matrix generated by m_0. For any bba m on Ω, the bba $m \cdot S_{m_0}$ is less committed than the other bba $m \cdot S$ that could be obtained by the use of the other specialization matrices S of Σ_0. Dempster's rule of combination satisfies that form of minimal committment.

Theorem 4: Let m_0 be a bba on Ω. Let $\Sigma_0 = \{S \in \Sigma \mid \forall m: m \cdot S \sqsubseteq m_0\}$.
Then $S_{m_0} \in \Sigma_0$ and $\forall S \in \Sigma_0, \forall m, m \cdot S \sqsubseteq m \cdot S_{m_0}$.

Proof. $S_{m_0} \in \Sigma_0$ holds, since $m \cdot S_{m_0} = m \oplus m_0 \sqsubseteq m_0$. Let $S_0 = S_{m_0}$. Let $S \in \Sigma_0$. For $A \subseteq \Omega$, we define the bba $m_S^{(A)}$ by $m_S^{(A)}(B) = s(A,B)$ with corresponding plausibilty function $pl_S^{(A)}$.
Let m be the bba with m(A)=1.
$m.S = m_S^{(A)}$ and therefore $m_S^{(A)} \sqsubseteq m_0$ implying
$m_S^{(A)} = m_S^{(A)}(.|A) \sqsubseteq m_0(.|A) = m_{S_0}^{(A)}$. (*)
Now let m be an arbitrary bba on Ω. Let pl' be the plausibilty function correponding to $m \cdot S$.

---

[1] The row vector t is an eigenvector of the matrix S if $t \, S = \lambda \, t$ where λ is a scalar.

$$pl'(A) = \sum_{B\cap A\neq\emptyset} \sum_{B\subseteq C} m(C)\, s(C,B) = \sum_{C\cap A\neq\emptyset} \sum_{B\subseteq C\ B\cap A\neq\emptyset} m(C)\, s(C,B)$$

$$= \sum_{C\cap A\neq\emptyset} m(C)\, pl_S^{(C)}(A) \leq \sum_{C\cap A\neq\emptyset} m(C)\, pl_{S_0}^{(C)}(A) = pl''(A)$$

where pl" denotes the plausibilty function corresponding to $m_0 \cdot S_0$ and the inequality results from (*). QED

8. Generalization to contraction of beliefs and disjunctive combinations.

The presentation centered around the expansion process. But it can be adapted in a direct way to the inverse process of contraction, i.e. the change in belief where a belief is eliminated from the evidential corpus EC. The specialization matrices encountered in the expansion processes are replaced by de-specialization matrices. The de-specialization matrices are the (matricial) inverse of the specialization matrices. In contrast with the specialization matrices, they should not be applied to any bba, but only to a subset of bba that depends on the de-specialization matrix. This constraint reflects the fact that one can not eliminate from EC an evidence that had not been previously added to EC.

Pieces of evidence are combined conjunctively in the expansion: the aim is to compute the belief function $bel_{E \text{ and } F}$ induced by the conjunction 'E and F' of two pieces of evidence E and F from the belief functions bel_E and bel_F induced by each piece of evidence. In the retraction, the process is also a form of conjunctive combination. Eliminating evidence E from the EC that contains E and F consists in adding the information 'eliminate E'. It aims at computing bel_F from $bel_{E \text{ and } F}$ and bel_E (in practice an easy task when commonality functions are considered as $\forall A\subseteq\Omega\ q_F(A) = q_{E \text{ and } F}(A) / q_E(A)$).

One can define the dual processes of disjunctive combination of evidence. The typical example is illustrated by the case where You know bel_E and bel_F, but you can add only 'E or F' to EC. You must established $bel_{E \text{ or } F}$ from bel_E and bel_F. The disjunctive rule of combination has been defined and justified in Smets (1991c), the major result being that the product of bbm $m_E(A)$ and $m_F(B)$ is transfered to $A\cup B$ in $m_{E \text{ or } F}$, and $\forall A\subseteq\Omega\ bel_{E \text{ or } F}(A) = bel_E(A)\, bel_F(A)$.

The dual of the specialization matrix is called a generalization matrix G. It is also a stochastic matrix where the elements g(A,B) are null $\forall A\not\subseteq B\subseteq\Omega$. It represents the fact that a bbm m(A) is distributed among the supersets of A. The dual of the conditioning process is called the enlargement process. Let m' be the enlargement on A of a bba m. Then the conditioning of m' on any subset $X\cup Y$ of Ω with $X\subsetneq\bar{A}$ and $Y\subseteq A$ will give the same result whatever $Y\subseteq A$. The enlargement transforms m into m' where the elements of A become 'undiscernable' (in that no subset of A receives a support that is not given to the whole A).

Table 1 summarizes the dual relations.

Finally it can also be shown that the dual and the inverse relations are in one-to-one correspondance. To any de-specialization matrix corresponds a generalization matrix, and vice versa. So we do not have four processes, but only two. Which of each pair of equivalent approaches is more convenient depends on the context.

evidence combination	conjunctive	disjunctive
from bel_E and bel_F	$bel_{E \text{ and } F}$	$bel_{E \text{ or } F}$
expansion matrices	specialization	generalization
retraction matrices	de-specialization	de-generalization
single fact	conditioning	enlargement
	de-conditioning	de-enlargement
combination	conjunctive rule	disjunctive rule

9. Conclusions.

We provide justifications for Dempster's rule of conditioning and Dempster's rule of combination based on the following assumptions:
1. the expansion of any evidential corpus EC by an evidence E induces an updating from a prior bba m_0 into a postrior bba m_1 such that m_1 is a specialization of m_0.
2. if the evidence E says that all elements in $\overline{A}\subseteq\Omega$ are impossible, then the specialization m_1 should be such that $pl_1(\overline{A})=0$, and the least comitted specialization m_1 is the one obtained by applying Dempster's rule of conditioning on m_0.
3. the result of the expansion of Your EC by two pieces of evidence should be independent of the order with which the two pieces of evidence are taken in consideration. Therefore commutativity is required for the specialization matrices that represent these expansions, including those describing the conditioning process. The only specialization matrices satisfying these commutativity requirements are such that the result of their application is the same as the one obtained by the application of Dempster's rule of combination.

Acknowledgement.
This research has been supported by the ESPRIT II Basic Research Project 3085 (DRUMS) which is funded by the European Communities.

Bibliography.

DELGADO M. and MORAL S. (1987) On the concept of possibility-probabilty consistency. Fuzzy Sets and Systems 21: 311-3018.
DUBOIS D. and PRADE H. (1986) A set theoretical view of belief functions. Int. J. Gen. Systems, 12:193-226.
DUBOIS D. and PRADE H. (1987) The principle of minimum specificity as a basis for evidential reasoning. in: Uncertainty in knowledge-based systems, BOUCHON B. and YAGER R. eds, Springer Verlag, Berlin, pg.75-84.
DUBOIS D. and PRADE H. (1986) On the unicity of Dempster's rule of combination. Int. J. Intell. Systems 1:133-142.

GARDENFORS P. (1988) Knowledge in flux. Modelling the dynamics of epistemùic states. MIT Press, Cambridge, Mass.
GOODMAN I.R., NGUYEN H.T. and WALKER E.A. (1991) Conditional inference and logic for intelligent systems. Elsevier, Amsterdam.
GOODMAN I.R., GUPTA M.M., NGUYEN H.T. and ROGERS G.S. (1991) Conditional logic in expert systems. Elsevier, Amsterdam.
HSIA Y.-T. (1991) Characterizing Belief with Minimum Commitment. IJCAI-91:1184-1189.
KENNES R.(1991) Evidential Reasoning in a Categorial Perspective : Conjunction and Disjunction of Belief Functions" TR/IRIDIA/91-6
KLAWONN F. and SCHWECKE E. (1990) On the axiomatic justification of Dempster's rule of combination. (to appear in Int. J. Intell. Systems)
KRUSE R. and SCHWECKE E. (1990) Specialization: a new concept for uncertainty handling with belief functions. Int. J. Gen. Systems 18:49-60.
KYBURG H.E.Jr. (1987) Bayesian and non-Bayesian evidential updating. Artificial Intelligence, 31:271-294.
LEVI I.(1980) The enterprise of knowledge. MIT Press, Cambridge, Mass.
LEVI I. (1983) Consonance, dissonance end evidentiary mechanisms. in GARDENFORS P., HANSSON B. and SAHLIN N.E. (eds) Evidentiary value: philosophical, judicial and psychological aspects of a theory. C.W.K. Gleerups, Lund, Sweden. p. 27-43.
MORAL S. (1985) Informacion difusa. Relationes eentre probabilidad y possibilidad. Tesis Doctoral, Universidad de Granada.
NGUYEN T. H. and SMETS Ph. (1991) On Dynamics of Cautious Belief and Conditional Objects. TR/IRIDIA/91-13. Submitted for Publication.
PEARL J. (1990) Reasoning with Belief Functions: an Analysis of Compatibility. Intern. J. Approx. Reasoning, 4:363-390.
SHAFER G. (1976) A mathematical theory of evidence. Princeton Univ. Press. Princeton, NJ.
SMETS Ph. (1988) Belief functions. in SMETS Ph, MAMDANI A., DUBOIS D. and PRADE H. eds. Non standard logics for automated reasoning. Academic Press, London p 253-286.
SMETS Ph. (1990) The combination of evidence in the transferable belief model. IEEE-Pattern analysis and Machine Intelligence, 12:447-458.
SMETS Ph. (1991a) Resolving misunderstandings about belief functions: A response to the many criticisms raised by J. Pearl. To appear in Int. J. Approximate Reasoning.
SMETS Ph. (1991b) What is Dempster-Shafer model? TR/IRIDIA/91-20 (submitted for publication)
SMETS Ph. (1991c) Belief Functions : the Disjunctive Rule of Combination and the Generalized Bayesian Theorem. TR/IRIDIA/91-1
SMETS Ph. and KENNES (1990) The transferable belief model. Technical Report: TR-IRIDIA-90-14.
VOORBRACHT F. (1991) On the justification of Dempster's rule of combination. Artificial Intelligence 48:171-197.
YAGER R. (1986) The entailment principle for Dempster-Shafer granules. Int. J. Intell. Systems 1:247-262

Let ECØ denote the empty (vacuous) evidential corpus, i.e. the one obtained in the absence of any evidence (or at least of any evidence relevant to Your belief on the frame of discernment Ω under consideration). Let m_0 be the bba induced by Your EC on a given frame of discernment Ω.

We say that E belongs to EC if EC$\models$E.