The α-junctions: the commutative and associative non interactive combination operators applicable to belief functions.

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1. Introduction.

In the transferable belief model (TBM), the classical and well-known combination rule is the so-called Dempster's rule of combination (for the TBM, see Smets and Kennes, 1994, for Dempster's rule of combination, see Shafer, 1976, Smets, 1990). This rule corresponds to a conjunction operator: it builds the belief induced by accepting two pieces of evidence, i.e., by accepting their conjunction. Besides there also exists a disjunctive rule of combination (Smets, 1993a). Finally, there is still a third rule, usually forgotten, that fits with the exclusive disjunction. When we noticed this third rule, we came to the idea that these three rules may be special cases of a more general combination scheme... and discovered what we will call the α-junction rules. These new rules could be extended for combining weighted sets, nevertheless our presentation is restricted to the domain covered by the TBM, i.e., to belief functions.

Conceptually what is a belief function within the TBM? It is a function that quantifies the strength of the beliefs held by a given agent, called You, at a given time t. We assume a set Ω of possible worlds, one of them is the actual world and we denote it ω_0. You, the agent, do not know exactly which world in Ω is ω_0 and all You can express is the strength of Your belief that $\omega_0 \in A$, for every $A \subseteq \Omega$. This strength is quantified by a belief function $bel: 2^\Omega \rightarrow [0,1]$ with bel(A) representing the strength of Your belief that the actual world ω_0 belongs to the subset A of Ω.

These strengths result from the pieces of evidence relative to ω_0 that You have accumulated. What is a piece of evidence? Suppose a source of information, denoted S, that states that a proposition E is true and You accept at time t that S is telling the truth. We call this whole fact a piece of evidence, and we denote it $\mathtt{E}$. So a piece of evidence $\mathtt{E}$ is a triple (S, E, true) where S is a source of information, E is the proposition states by S, and true denotes that you accept as true what the source states. To be complete 'You' and 't' should also be included, but we neglect them as they stay constant all over this presentation.

This definition is not exactly the same as 'accepting E', as it will be seen once negation is introduced. Suppose S states that E is true and You accept at time t that S is telling the false (so S is lying). We define this piece of evidence as the negation of $\mathtt{E}$. It is the triple (S, E, false) and we denote it $\sim\mathtt{E}$. Its meaning will become clearer once the α-junctions will have been studied. Intuitively, it seems acceptable to defend that $\sim\mathtt{E}$ is equivalent to: S states that $\neg$E is true and You accept at time t that S is telling the truth, i.e., (S, E, false) = (S, ¬E, true). But if we had defended that $\mathtt{E}$ is ‘accepting E’, than $\mathtt{\sim E}$ would have been understood as ‘not accepting E’, whereas it is closer to ‘accepting ¬E’. So using the modal operator ‘accepting’ is not adequate here.

Why to distinguish between $\mathtt{E}$ and E? Suppose two sources S_1 and S_2, and S_1 states E_1 and S_2 states S_2. Suppose You accept at t that at least one of S_1 or S_2 is telling the truth. This is denoted here as $\mathtt{E}_1 \vee \mathtt{E}_2$. If we had not distinguished between $\mathtt{E}$ and E, than we would have written $E_1 \vee E_2$. With such a notation, we could not distinguish the present situation with the following one: suppose the source S states that $E_1 \vee E_2$ is true and You accept at time t that S is telling the truth. In the first case, the sources are precise but You accept that maybe one of them is lying, whereas in the second case, You accept that the source tells the truth, but the source is not very precise. The first case is a problem of uncertainty (which source tells the truth), whereas the second is a case of imprecision (Smets, 1997). To furhter enhance the difference, suppose You want to better Your information. In the first case, You would worry about which source is telling the truth and collect information about the reliability of the sources. In the second case, You would worry directly about which proposition is true. In the context model, Gebhardt and Kruse (1993) also insist in taking in account the nature of the sources of information, and not only what they state.

Coming back to the two sources S_1 and S_2 where S_1 states E_1 and S_2 states E_2. They can be combined in three natural ways. (We use the same $\vee$, $\wedge$ and $\underline{\vee}$ operators for combining pieces of evidence as those used in classical logic to combine propositions. The symbol $\underline{\vee}$ denotes the exclusive disjunction operator.)

1. Suppose You accept at t that both S_1 and S_2 are telling the truth, what we denote by $\mathtt{E}_1 \wedge \mathtt{E}_2$. We call this combination a conjunctive combination or a conjunction of two pieces of evidence.
2. Suppose You accept at t that at least one of S_1 or S_2 is telling the truth, what we denote by $\mathtt{E}_1 \vee \mathtt{E}_2$. We call this combination a disjunctive combination or a disjunction of two pieces of evidence.
3. Suppose You accept at t that one and only one of S_1 or S_2 is telling the truth, what we denote by $\mathtt{E}_1 \underline{\vee} \mathtt{E}_2$. We call this combination an exclusive disjunctive combination or an exclusice disjunction of two pieces of evidence. (Note that in propositional logic, the exclusive disjunction $E_1 \underline{\vee} E_2$ is equivalent to $(E_1 \vee E_2) \wedge \neg(E_1 \wedge E_2)$)

Suppose now that $\mathtt{E}$ is the only piece of evidence that You have accumulated about which of the worlds in Ω is the actual world ω_0. $\mathtt{E}$ induces in You a belief function, denoted bel[$\mathtt{E}$], on Ω that represents Your beliefs defined on Ω at t about the value of ω_0. The basic belief assignment (bba) related to bel[$\mathtt{E}$] is denoted m[$\mathtt{E}$] and m[$\mathtt{E}$](A) denotes the basic belief mass (bbm) given to $A \subseteq \Omega$ by the bba m[$\mathtt{E}$].

Suppose two pieces of evidence $\mathtt{E}_1$ and $\mathtt{E}_2$. Let bel[$\mathtt{E}_1$] and bel[$\mathtt{E}_2$] be the belief functions on Ω that they would have induced individually.

1. Suppose You accept that both sources of evidence tell the truth, then You build the belief function bel[$E_1 \wedge E_2$] induced by the conjunction of E_1 and E_2. If we assume that this new belief function depends only on bel[E_1] and bel[E_2], what translates the idea that they are 'distinct' (Smets, 1992) or non interactive, then bel[$E_1 \wedge E_2$] is obtained by Dempster's rule of combination (unnormalized in this case). The bba m[$E_1 \wedge E_2$] satisfies:

$$m[E_1 \wedge E_2](A) = \sum_{X,Y \subseteq \Omega:\ X \cap Y = A} m[E_1](X)\ m[E_2](Y) \qquad \text{for all } A \subseteq \Omega$$

This rule is called hereafter the conjunctive rule of combination, as it results from the conjunction of the two pieces of evidence.

2. Now suppose instead that You accept that at least one source of evidence tells the truth, then You build the belief function bel[$E_1 \vee E_2$] induced by the disjunction of E_1 and E_2. You know what would be Your beliefs if You had known which source tells the truth, they are bel[E_1] and bel[E_2], respectively. But You are not so knowledgeable about E_1 and E_2 and You must limit Yourself in building bel[$E_1 \vee E_2$]. Just as Dempster's rule of combination fits the conjunctive case, the so-called disjunctive rule of combination solves the disjunctive case (Smets, 1993a). In that case the corresponding bba m[$E_1 \vee E_2$] satisfies:

$$m[E_1 \vee E_2](A) = \sum_{X,Y \subseteq \Omega:\ X \cup Y = A} m[E_1](X)\ m[E_2](Y) \qquad \text{for all } A \subseteq \Omega$$

3. One could also imagine the case where You accept that one and only one source of evidence tells the truth, but You don't know which one is telling the truth. This is the exclusive disjunction. So we build bel[$E_1 \underline{\vee} E_2$]. The bba m[$E_1 \underline{\vee} E_2$] satisfies:

$$m[E_1 \underline{\vee} E_2](A) = \sum_{X,Y \subseteq \Omega:\ X \underline{\cup} Y = A} m[E_1](X)\ m[E_2](Y) \qquad \text{for all } A \subseteq \Omega$$

where $\underline{\cup}$ is the symmetric difference, i.e., $X \underline{\cup} Y = (X \cap \overline{Y}) \cup (\overline{X} \cap Y)$.

These rules can in fact be extended to any number of pieces of evidence and any combination formula that states which source You accept as telling the true. So let E_1, E_2... E_n be a set of pieces of evidence, with bel[E_i], i=1,2...n be the belief functions induced by each piece of evidence individually. Suppose the pieces of evidence are non interactive, i.e., the belief function build from the combination of the pieces of evidence is a function of the belief functions bel[E_i]). For instance, suppose all You accept is that $(E_1 \wedge E_2) \vee E_3) \underline{\vee} (E_4 \wedge E_1)$ holds. It means You accept that one and only one of the two following cases holds: $(E_1 \wedge E_2) \vee E_3$ or $E_4 \wedge E_1$. In the first case, You accept that at least one of the next two cases holds: $(E_1 \wedge E_2)$ orE_3. It means that you accept that either S_1 and S_2 tell the truth or S_3 tells the truth, in a non exclusive way. In the second case, You accept that both S_4 and S_1 tell the truth. Given this complex piece of evidence, the basic belief masses related to the belief function bel[$((E_1 \wedge E_2) \vee E_3) \underline{\vee} (E_4 \wedge E_1)$] is:

$m[((E_1 \wedge E_2) \vee E_3) \underline{\vee} (E_4 \wedge E_1)](A) =$

$$\sum_{X,Y,Z,T\subseteq\Omega:\ ((X\cap Y)\cup Z)\underline{\cup}(T\cap X)=A} m[E_1](X)\ \ m[E_2](Y)\ \ m[E_3](Z)\ \ m[E_4](T)$$

for all $A\subseteq\Omega$

This result was known for long (e.g., Dubois and Prade, 1986). It covers of course the conjunctive rule, the disjunctive rule and the exclusive disjunctive rule, three particular cases where there are only two pieces of evidence. Discovering these three cases that can be built with two pieces of evidence, we came to the idea that these three cases are nothing but special cases of a more general combination rule and we have discovered, as shown here after, the existence of a parametrized family of combination rules (with one parameter), the three special cases corresponding to special values of the parameter. We have called this new family of combination rules, the α-junction where α is the parameter of the combination rule and -junction is the common part of both the 'conjunction' and 'disjunction' words.

The concept of negation, and its meaning, came out of our developement. Suppose the pieces of evidence E. Dubois and Prade (1986) have suggested that if bel[E] is the belief induced by E, then bel[~E] could be defined so that its bba satisfies:

$$m[\sim E]\ (A) = m[E]\ (\overline{A}) \qquad \text{for all } A\subseteq\Omega$$

and where $\overline{A}$ is the complement of A relative to Ω (Dubois and Prade, 1986, have used the notation $\overline{m}$ for m[~E]). This definition will be used later when we will study the De Morgan properties of the α-junctions.

We have thus found out the conjunction, disjunction and exclusive disjunction operators, and the negation. Readers might wonder if there are not other symmetrical junctions operators that can be built from two propositions in classical logic. In fcat, there are only eight symmetrical operators that can be built with two propositions: the tautology, the conjunction, the disjunction, and the exclusive disjunction, and their negations, the contradiction, the disjunction of the negations, the conjunction of the negations, and a junction without name. Figure 1 shows these eight operators. Two elements opposed by a diagonal are the negation of each other.

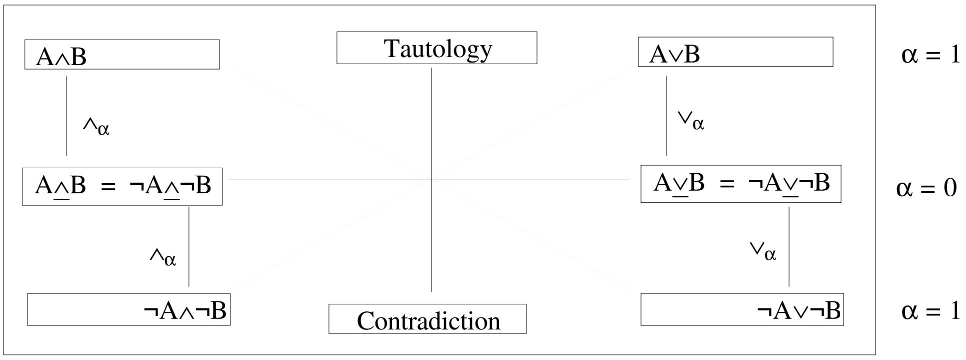

Figure 1: The eight symmetrical junctions operaotrs in classical logic, where $\underline{\vee}$ denotes the exclusive disjunction and $\underline{\wedge}$ denotes its $\neg$ dual ($A\underline{\wedge}B = \neg(A\underline{\vee}B) = (A\wedge B)\vee(\neg A\wedge\neg B)$). Diagonaly opposed pairs are linked by the negation operator $\neg$. The four vertical lines at left and right are the forthcoming α-conjunction and α-disjunction operaotrs.

In this paper, we present first some needed definitions and notation conventions. We proceed by studying the α-junctive rule of combination of two pieces of evidence. Then we study the disjunctive and the conjuncitve cases, and conclude.

2. Definitions and notations.

2.1. Belief functions

A basic belief assignment (bba) is defined as the function from 2^Ω to [0,1], its values are the basic belief masses (bbm) and their sum over the subsets of Ω is 1. To simplify the notation we write m_1, m_2... for $m[E_1]$, $m[E_2]$... and even drop the reference to the underlying piece of evidence when it is irrelevant to the presentation.

In the TBM, the mass m(A) for $A\subseteq\Omega$ is that part of Your belief that supports that the actual world ω_0 is in A and nothing more specific. The belief function bel is defined as

$$bel(A) = \sum_{B\subseteq\Omega:\emptyset\neq B\subseteq A} m(B).$$

It represents the total belief that supports that the actual world ω_0 is in A.

Related to m and bel, the commonality function $q:2^\Omega\to[0,1]$ is defined as :

$$q(A) = \sum_{B\subseteq\Omega:A\subseteq B} m(B)$$

and we introduce the function $b:2^\Omega\to[0,1]$ defined as:

$$b(A) = \sum_{B\subseteq\Omega:\ B\subseteq A} m(B) = bel(A) + m(\emptyset).$$

The meaning of q and b is essentially technical, even though q(A) can be understood as the ignorance about ω_0 when You know that ω_0 belongs to A. Their major use is to be found in the combination rules. Given two bba m_i, i= 1,2, with q_i and b_i their related q- and b-functions, then $q_{1\wedge 2}$ and $b_{1\vee 2}$, the q and b-functions that result from their combinations, are given by:

in the conjunctive case: $q_{1\wedge 2}(A) = q_1(A)\, q_2(A)$ for all $A\subseteq\Omega$

in the disjunctive case: $b_{1\vee 2}(A) = b_1(A)\, b_2(A)$ for all $A\subseteq\Omega$.

Besides $b[\sim E](A) = q[E](\overline{A})$ and $q[\sim E](A) = b[E](\overline{A})$ for all $A\subseteq\Omega$, a property that fits with De Morgan law. Indeed the combination rules can be written as:

$$q[E_1\wedge E_2](A) = q[E_1](A)\, q[E_2](A)$$

and $$b[E_1\vee E_2](A) = b[E_1](A)\, b[E_2](A).$$

Then: $$b[\sim(E_1\wedge E_2)](A) = q[E_1\wedge E_2](\overline{A}) = q[E_1](\overline{A})\, q[E_2](\overline{A})$$
$$= b[\sim E_1](A)\, b[\sim E_2](A) = b[\sim E_1\vee\sim E_2](A)$$

So: $b[\sim(E_1\wedge E_2)](A) = b[\sim E_1\vee\sim E_2](A)$ for all $A\subseteq\Omega$,

Similarly,

$q[\sim(E_1\vee E_2)](A) = q[\sim E_1\wedge\sim E_2](A)$ for all $A\subseteq\Omega$.

These two relations are the De Morgan formulas as they show that $\sim(E_1\wedge E_2)$ and $\sim E_1\vee\sim E_2$ induce the same bba's (and identically for $\sim(E_1\vee E_2)$ and $\sim E_1\wedge\sim E_2$).

2.2. Notation.

A bba m defined on Ω can be represented as a vector with $2^{|\Omega|}$ elements, so $\mathbf{m} = [m(X)]$ where X is the line index of the component of the vector **m** and $X\subseteq\Omega$. The order of the elements in **m** is arbitrary, but one order is particularly convenient as it enhances many symmetries. This order is a kind of lexico-iterated order. E.g. let $\Omega = \{a,b,c\}$, then the transpose **m**' of the vector **m** is given by:

m' = (m(Ø), m({a}), m({b}), m({a,b}, m({c}), m({a,c}), m({b,c}), m([a,b,c})).

All matrices and vectors in this paper will be organized so that their indices obey to this order.

We use the notation 1_X to denoted a bba where all elements are null except the X'th element that equals 1: it is the bba that gives a mass 1 to the set X.

We also use the following notations:

1 is a vector where all elements equal to 1.

I is the identity matrix.

J is the matrix with elements j_{XY} where $j_{XY} = 1$ if $X = \overline{Y}$, and $j_{XY} = 0$ otherwise. With $\Omega = \{a,b\}$,

$$\mathbf{J} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

The **J** matrix is the operator that transforms a bba m into its negation: $\mathbf{J}.\mathbf{m}[E] = \mathbf{m}[-E]$. Indeed, **J** projects m(A) on $\overline{A}$ for all $A\subseteq\Omega$. We also have $\mathbf{J}.\mathbf{J} = \mathbf{I}$, what corresponds to the involutive property of the negation operator.

Given a vector $\mathbf{v}$, [diag $\mathbf{v}$] is the diagonal matrix whose diagonal elements are the values of $\mathbf{v}$, all other elements being 0.

2.3. Permutation.

Let P be a permutation from Ω to Ω. Let $P(X) = \{y: y = P(x), x \in X\}$. We define as $\mathbf{L}_P$ the permutation matrix from 2^Ω to 2^Ω obtained from the permutation P, and such that it maps the element $X \subseteq \Omega$ onto the element $P(X) \subseteq \Omega$. With $\Omega = \{a,b\}$ and P such that $P(a) = b$ and $P(b) = a$, $\mathbf{L}_P$ is:

$$\mathbf{L}_P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Lemma 1: $P(A) = A$ for all P defined on Ω iff $A = \emptyset$ or $A = \Omega$.

The only subsets of Ω that are mapped onto themselves whatever the permutation are $\emptyset$ and Ω. Permutation matrix also satisfies $(\mathbf{L}_P)^{-1} = \mathbf{L}_{(P^{-1})}$ (hence the parenthesis can be dropped without the risk of a typographical confusion).

2.4. Combination rules in matrix notation.

The three combinations rules can be represented under matrix forms. The conjunctive combination is introduced in Klawonn and Smets (1992). Suppose $\Omega = \{a,b\}$. Let $\mathbf{D}(\mathbf{m})$ be the following matrix:

$$\mathbf{D}(\mathbf{m}) = \begin{bmatrix} 1 & m(\emptyset)+m(b) & m(\emptyset)+m(a) & m(\emptyset) \\ 0 & m(a)+m(\Omega) & 0 & m(a) \\ 0 & 0 & m(b)+m(\Omega) & m(b) \\ 0 & 0 & 0 & m(\Omega) \end{bmatrix}$$

Then the bba $\mathbf{m}_1 = \mathbf{D}(\mathbf{m}).\mathbf{m}_0$ is equal to the bba one would obtained by combining $\mathbf{m}_0$ and $\mathbf{m}$ by the conjunctive rule of combination (i.e., Dempster's rule of combination, but unnormalized). In general, the A,B element of $\mathbf{D}(\mathbf{m})$ is $m(A|B) = \sum_{X \subseteq \Omega:\, X \cap B = A} m(A \cup X)$, i.e., the B'th column is the vector obtained by conditioning $\mathbf{m}$ on B, or equivalently conjunctively combining $\mathbf{m}$ with the bba $\mathbf{1}_B$.

Similar results are derivable for the disjunction combination described in Smets (1993a). Let $\mathbf{E}(\mathbf{m})$ be the following matrix:

$$\mathbf{E}(\mathbf{m}) = \begin{bmatrix} m(\emptyset) & 0 & 0 & 0 \\ m(a) & m(a)+m(\emptyset) & 0 & 0 \\ m(b) & 0 & m(b)+m(\emptyset) & 0 \\ m(\Omega) & m(b)+m(\Omega) & m(a)+m(\Omega) & 1 \end{bmatrix}$$

Then the bba $\mathbf{m}_1 = \mathbf{E}(\mathbf{m}).\mathbf{m}_0$ is equal to the bba one would obtained by combining $\mathbf{m}_0$ and $\mathbf{m}$ by the disjunctive rule of combination. In general, the A,B element of $\mathbf{E}(\mathbf{m})$ is $\sum_{X\subseteq\Omega:\ X\cup B=A} m(A\cup X)$, i.e., the B'th column is the vector obtained by disjunctively combining $\mathbf{m}$ with the bba $\mathbf{1}_B$.

For the exclusive disjunction combination, the matrix is given by:

$$\mathbf{F}(\mathbf{m}) = \begin{bmatrix} m(\emptyset) & m(a) & m(b) & m(\Omega) \\ m(a) & m(\emptyset) & m(\Omega) & m(b) \\ m(b) & m(\Omega) & m(\emptyset) & m(a) \\ m(\Omega) & m(b) & m(a) & m(\emptyset) \end{bmatrix}.$$

It can be verified that the bba $\mathbf{m}_1 = \mathbf{F}(\mathbf{m}).\mathbf{m}_0$ is indeed the bba one would obtained by combining $\mathbf{m}_0$ and $\mathbf{m}$ by the exclusive disjunction.

3. The α-junctions.

3.1. The matrix K(m).

Let $\mathbf{m}_1$ and $\mathbf{m}_2$ be two basic belief assignments on Ω. We assume that there exists an operator $[K(\mathbf{m}_1)]$ induced by $\mathbf{m}_1$ so that, when applied to $\mathbf{m}_2$, it produces a combination $\mathbf{m}_{12}$ of $\mathbf{m}_1$ with $\mathbf{m}_2$.

$$\mathbf{m}_{12} = [K(\mathbf{m}_1)]\ \mathbf{m}_2$$

The first step consists in showing why $[K(\mathbf{m}_1)]$ is a linear operator. Suppose three bba $\mathbf{m}_0$, $\mathbf{m}_1$ and $\mathbf{m}_2$. Let $[K(\mathbf{m}_0)]$ be the operator induced by $\mathbf{m}_0$. Suppose $\mathbf{m}_1$ ($\mathbf{m}_2$) is the bba that would describe Your beliefs if You accept that S_1 (S_2) tells the truth: $\mathbf{m}_1 = \mathbf{m}[E_1]$ and $\mathbf{m}_2 = \mathbf{m}[E_2]$. It happens You know that one of E_1 or E_2 will be accessible to You. Which one will be decided by a random device (such as tossing a coin). In case of success, E_1 will be the piece of evidence You will hold, otherwise E_2 will be the piece of evidence You will hold. Let p be the probability that a success occurs and q = 1 - p. Before knowing the outcome of this random experiment, Your bba is $\mathbf{m}_{12} = p.\mathbf{m}_1 + q.\mathbf{m}_2$ (for a justification of this linear relation, see Smets, 1993b). Consider the results of the combination of $[K(\mathbf{m}_0)]$ with $\mathbf{m}_1$ and $\mathbf{m}_2$ individually. We postulate that before knowing the outcome of the random experiment, the result of combining $[K(\mathbf{m}_0)]$ to $\mathbf{m}_{12}$ would be equal to the same linear combination of $[K(\mathbf{m}_0)]\mathbf{m}_1$ and $[K(\mathbf{m}_0)]\mathbf{m}_2$. We assume that combining and averaging commute.

Assumption A1: Linearity.

$$[K(\mathbf{m}_0)](p.\mathbf{m}_1 + q.\mathbf{m}_2) = p.[K(\mathbf{m}_0)]\mathbf{m}_1 + q.[K(\mathbf{m}_0)]\mathbf{m}_1$$

This assumption is sufficient to conclude that $[K(\mathbf{m}_0)]$ is a linear operator and can thus be represented by a matrix that we denoted by $\mathbf{K}(\mathbf{m}_0)$. So the operation $[K(\mathbf{m}_0)]\mathbf{m}_1$ is nothing but the matricial product of $\mathbf{K}(\mathbf{m}_0)$ with the vector $\mathbf{m}_1$.

We next assume that the combination of $\mathbf{m}_1$ and $\mathbf{m}_2$ commute, i.e., combining $\mathbf{m}_1$ with $\mathbf{m}_2$ or $\mathbf{m}_2$ with $\mathbf{m}_1$ leads to the same result.

Assumption 2 : Commutativity.

$$\mathbf{K}(\mathbf{m}_0)\ \mathbf{m}_1 = \mathbf{K}(\mathbf{m}_1)\ \mathbf{m}_0$$

Theorem 1: Under assumptions A1 and A2,

$$\mathbf{K}(\mathbf{m}) = \sum_{X \subseteq \Omega} m(X)\ \mathbf{K}_X.$$

where the $\mathbf{K}_X$ matrices are matrices which coefficients do not depend on **m.**

Proof:

By A1, $\mathbf{K}(\mathbf{m}_0)\ (p.\mathbf{m}_1 + q.\mathbf{m}_2) = p.\mathbf{K}(\mathbf{m}_0)\ \mathbf{m}_1 + q.\mathbf{K}(\mathbf{m}_0)\ \mathbf{m}_2.$

By A2, $\mathbf{K}(\mathbf{m}_0)\ (p.\mathbf{m}_1 + q.\mathbf{m}_2) = \mathbf{K}(p.\mathbf{m}_1 + q.\mathbf{m}_2)\ \mathbf{m}_0$

$\mathbf{K}(\mathbf{m}_0)\ \mathbf{m}_1 = \mathbf{K}(\mathbf{m}_1)\ \mathbf{m}_0$

$\mathbf{K}(\mathbf{m}_0)\ \mathbf{m}_2 = \mathbf{K}(\mathbf{m}_2)\ \mathbf{m}_0$

This being true whatever $\mathbf{m}_0$, we get:

$$\mathbf{K}(p.\mathbf{m}_1 + q.\mathbf{m}_2) = p\ \mathbf{K}(\mathbf{m}_1) + q\ \mathbf{K}(\mathbf{m}_2)$$

It implies that is linear in m, thus the theorem. QED

From A2 we can also derive another constraint that the $\mathbf{K}_X$ matrices must satisfy . Let $\mathbf{K}_X = [k^X_{AB}]$ where A, B⊆Ω. So k^X_{AB} denotes the element of $\mathbf{K}_X$ at line A and column B.

Theorem 2: $k^X_{AY} = k^Y_{AX}$ for all A,X,Y⊆Ω.

Proof: The requirement

$$\mathbf{K}(\mathbf{m}_1)\ \mathbf{m}_2 = \mathbf{K}(\mathbf{m}_2)\ \mathbf{m}_1 \tag{3.1}$$

becomes for A⊆Ω,

$$\sum_{X \subseteq \Omega} \mathbf{m}_1(X) \sum_{Y \subseteq \Omega} k^X_{AY}.\mathbf{m}_2(Y) = \sum_{Y \subseteq \Omega} \mathbf{m}_2(Y) \sum_{X \subseteq \Omega} k^Y_{AX}.\mathbf{m}_1(X) \tag{3.2}$$

Being true whatever $\mathbf{m}_1$ and $\mathbf{m}_2$, one has $k^X_{AY} = k^Y_{AX}$ for all A,X,Y⊆Ω. QED

So the Y-th column of $\mathbf{K}_X$ is equal to the X-th column of $\mathbf{K}_Y$.

3.2. $\mathbf{K}_X$ is a stochastic matrix.

Theorem 3: For all $X \subseteq \Omega$, $\mathbf{K}_X$ is a stochastic matrix.

Proof: Suppose the following bba: $\mathbf{m}_1 = \mathbf{1}_X$ and $\mathbf{m}_2 = \mathbf{1}_Y$. Then (3.1) becomes:

$$\mathbf{K}(\mathbf{m}_1)\,\mathbf{m}_2 = \mathbf{K}(\mathbf{1}_X)\,\mathbf{1}_Y = k^X_{AY}$$

So the resulting bba is the column vector with elements k^X_{AY} for $A \subseteq \Omega$. Being a bba, its elements must be non negative and add to 1.

$$k^X_{AY} \geq 0 \qquad \text{and} \qquad \sum_{A \subseteq \Omega} k^X_{AY} = 1$$

QED

Thus each column of $\mathbf{K}_X$ can be assimilated to a probability distribution function over 2^Ω (in fact each column is a bba).

3.3. Anonymity.

Let P be a permutation of the elements of Ω. Let $\mathbf{L}_P$ be the permutation matrix as defined in section 2.3. When applied to a bba $\mathbf{m}$, $\mathbf{L}_P$ produces a new bba $\mathbf{m}_P$ that differs only from $\mathbf{m}$ by the fact that, for every A in Ω, the mass initially given to A is given after permutation to P(A).

For instance let $\Omega = \{a,b\}$ and $P:\Omega \to \Omega$ so that $P(a) = b$, $P(b) = a$. Then:

$$\mathbf{L}_P\,\mathbf{m} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m(\emptyset) \\ m(a) \\ m(b) \\ m(a,b) \end{bmatrix} = \begin{bmatrix} m(\emptyset) \\ m(b) \\ m(a) \\ m(a,b) \end{bmatrix}.$$

We assume that a renaming of the elements Ω will not affect the results of the combination.

Assumption 3 : Anonymity.
Let P be a permutation of Ω to Ω and let $\mathbf{L}_P$ be the permutation matrix that permutes the subset A into the subset P(A). Then

$$\mathbf{K}(\mathbf{L}_P\mathbf{m}_1)\,\mathbf{L}_P\mathbf{m}_2 = \mathbf{L}_P\mathbf{K}(\mathbf{m}_1)\mathbf{m}_2. \tag{3.3}$$

This assumption translates the following idea. Suppose we permute the elements of Ω in both $\mathbf{m}_1$ and $\mathbf{m}_2$, then the result of the combination is nothing but the permutation of the results of the combination of $\mathbf{m}_1$ with $\mathbf{m}_2$.

3.4. Symmetry.

Theorem 4: $\mathbf{K}_{P(X)}\,\mathbf{L}_P = \mathbf{L}_P\,\mathbf{K}_X$

Proof: Suppose $\mathbf{m}_1 = \mathbf{1}_X$. Then $\mathbf{L}_P\mathbf{1}_X = \mathbf{1}_{P(X)}$. Replace $\mathbf{m}_1$ by $\mathbf{1}_X$ in (3.3) and note that it is true for all $\mathbf{m}_2$.

QED

3.5. Vacuous belief.

We assume the existence of a bba (denoted $\mathbf{m}_{vac}$) which combination with any bba leaves it unchanged, i.e., a neutral element for the combination.

Assumption A4. Vacuous belief.
There exists a bba $\mathbf{m}_{vac}$ so that for any bba $\mathbf{m}$, $\mathbf{K}(\mathbf{m})\ \mathbf{m}_{vac} = \mathbf{m}$.

Theorem 5: $\mathbf{K}(\mathbf{m}_{vac}) = \mathbf{I}$.

Proof: By A2, A4 implies: $\mathbf{K}(\mathbf{m}_{vac})\ \mathbf{m} = \mathbf{m}$ for all $\mathbf{m}$, hence the theorem. QED

3.6. Associativity.

We assume that the combination is associative. This property means that the order with which the bba are combined is irrelevant.

Assumption A5: Associativity.
Let $\mathbf{m}_1$, $\mathbf{m}_2$ and $\mathbf{m}_3$ be three bba on Ω. Then:

$$\mathbf{K}(\mathbf{m}_1)\ (\mathbf{K}(\mathbf{m}_2)\mathbf{m}_3) = \mathbf{K}(\mathbf{K}(\mathbf{m}_1)\mathbf{m}_2)\ \mathbf{m}_3.$$

Theorem 6: $\mathbf{K}_X\mathbf{K}_Y = \mathbf{K}(\mathbf{K}_X\mathbf{1}_Y)$ for all $X,Y\subseteq\Omega$.

Proof: Let $\mathbf{m}_1 = \mathbf{1}_X$ and $\mathbf{m}_2 = \mathbf{1}_Y$. From A5, we get:

$$\mathbf{K}_X\ (\mathbf{K}_Y\mathbf{m}_3) = \mathbf{K}(\mathbf{K}_X\mathbf{1}_Y)\mathbf{m}_3.$$

This being true for any $\mathbf{m}_3$, thus the theorem. QED

Theorem 7: There exists an $X\subseteq\Omega$ so that $\mathbf{K}_X = \mathbf{I}$.

Proof: By theorem 5, we have: $\sum_{X\subseteq\Omega} m_{vac}(X)\ k^X_{AA} = 1.$

As $k^X_{AA}\in[0,1]$ (theorem 3), so $\sum_{X\subseteq\Omega} m_{vac}(X)\ k^X_{AA}$ is a weighted average of k^X_{AA} which values are also in [0,1]. The only way to get a sum equal to 1 is:
Case 1. $k^X_{AA}=1$ for all X (and all A), in which case $\mathbf{K}_X = \mathbf{I}$ for all X, and thus $\mathbf{K}(\mathbf{m}) = \mathbf{I}$, a degenerated (and uninteresting) solution that will be rejected by theorem 8.
Case 2. $m_{vac}(B) = 1$ for some $B\subseteq\Omega$ and the other values of $\mathbf{m}_{vac}$ are null. Then $k^B_{AA} = 1$ for all $A\subseteq\Omega$. As $\mathbf{K}_B$ is a stochastic matrix, $k^B_{AC} = 0$ if $A\neq C$, so $\mathbf{K}_B = \mathbf{I}$. QED

3.7. Reversibility.

We assume that different bba induce different operators.

Theorem 8. Reversibility. Let $\mathbf{m}_1$ and $\mathbf{m}_2$ be two bba on Ω.

If $\mathbf{m}_1 \neq \mathbf{m}_2$, then $\mathbf{K}(\mathbf{m}_1) \neq \mathbf{K}(\mathbf{m}_2)$.

Proof: Let $\mathbf{m}_1 \neq \mathbf{m}_2$, and suppose $\mathbf{K}(\mathbf{m}_1) = \mathbf{K}(\mathbf{m}_2)$. In that case, $\mathbf{K}(\mathbf{m}_1)\ \mathbf{m}_{vac} = \mathbf{K}(\mathbf{m}_2)\ \mathbf{m}_{vac}$, hence, by assumption A4: $\mathbf{m}_1 = \mathbf{m}_2$, contrary to the initial assumption. So the theorem. QED

This is just an assumption of reversibility for the **K** operator. It implies that $\mathbf{K}_X \neq \mathbf{K}_Y$ if $X \neq Y$ (take $\mathbf{m}_1 = \mathbf{1}_X$ and $\mathbf{m}_2 = \mathbf{1}_Y$). It eliminates also the degenerated solution (theorem 7, case 1) for the $\mathbf{m}_{vac}$ determination.

Theorem 9: $\mathbf{m}_{vac} = \mathbf{1}_{\emptyset}$ or $\mathbf{m}_{vac} = \mathbf{1}_{\Omega}$.

Proof: Consider now the $\mathbf{K}_B = \mathbf{I}$ and $\mathbf{m}_{vac} = \mathbf{1}_B$ (theorem 7, case 2). Let P be any permutation of the element of Ω, we have by construction

$$\mathbf{K}_{P(B)} = \mathbf{L}_P^{-1}\ \mathbf{K}_B\ \mathbf{L}_P = \mathbf{L}_P^{-1}\ \mathbf{L}_P = \mathrm{I}.$$

Thus $\mathbf{K}_{P(B)} = \mathbf{K}_B$ for all P, and this means that B is either Ø or Ω (lemma 1). QED

We have just rediscovered the existence of two vacuous bba that are well known in the TBM. Indeed, $\mathbf{1}_{\Omega}$ is the classical vacuous belief function of the TBM, the one initially described by Shafer and the one commonly called the vacuous belief function. It is the neutral element of the conjunctive rule of combination.

The other solution $\mathbf{1}_{\emptyset}$ for the vacuous bba is the negation of the previous solution. It is the neutral element of the disjunctive rule of combination (Smets, 19893a).

We call $\mathbf{1}_{\Omega}$ the and-vacuous bba and $\mathbf{1}_{\emptyset}$ the or-vacuous bba. In section 4, we will study in details the or-vacuous bba, hence the familly of disjuncitve combinations. All results obtained with it will be extended to the conjunctive case by an appropriate use of the negation operator and of the De Morgan formula (section 5).

3.8. Focused bba.

Suppose a bba on Ω so that:

$m(X) \geq 0$ if $X \subseteq A$,

$m(X) = 0$ otherwise.

This bba corresponds to the case where You know (fully believe) that the actual world ω_0 belongs to A. We will say that such a bba is focused on A. In such a case, Your beliefs would be the same if You had built them on A instead of Ω.

Suppose the two bba $\mathbf{m}_1$ and $\mathbf{m}_2$ are focused on A, then we assume that their combination is also focused on A. Once a world is 'eliminated' by both $\mathbf{m}_1$ and $\mathbf{m}_2$, it stays 'eliminated' after their combination.

Assumption A6. Context preservation.
Let $\mathbf{m}_1$ and $\mathbf{m}_2$ be two bba's on Ω so that:

$m_1(X) = m_2(X) = 0$ for all $X \not\subseteq A$,

then $(\mathbf{K}(\mathbf{m}_1)\ \mathbf{m}_2)(X) = 0$ for all $X \not\subseteq A$.

Theorem 10: $k^X_{AB} = 0$ for all $A \not\subseteq X$, $B \subseteq X \subseteq \Omega$.

Proof: Immediate. QED

3.9. Summary:

In summary, we have derived that $\mathbf{K}$ must satisfy:

P1:	$\mathbf{K}(\mathbf{m}) = \sum_{X \subseteq \Omega} m(X)\ \mathbf{K}_X$	Linearity
P2:	$k^X_{AY} = k^Y_{AX}$	Symmetry from commutativity
P3:	$k^X_{AB} \geq 0,\ \sum_{A \subseteq \Omega} k^X_{AB} = 1$	Stochastic matrix.
P4:	$\mathbf{K}_{P(X)} = \mathbf{L}_P\text{-}1\ \mathbf{K}_X\ \mathbf{L}_P$	Symmetry from anonymity
P5:	$\mathbf{K}(\mathbf{K}_X \mathbf{1}_Y) = \mathbf{K}_X \mathbf{K}_Y = \mathbf{K}_Y \mathbf{K}_X$	Associativity
P6:	$k^X_{AB} = 0$ for all $A \not\subseteq X$, $B \subseteq X$	Context preservation
P7:	$\mathbf{K}_\varnothing = \mathbf{I}$, $\mathbf{m}_{vac} = \mathbf{1}_\varnothing$ (disjunctive case) or	or-vacuous bba
	$\mathbf{K}_\Omega = \mathbf{I}$, $\mathbf{m}_{vac} = \mathbf{1}_\Omega$ (conjunctive case)	and-vacuous bba

4. The α-disjunctive combination.

In this section, we assume that the vacuous bba is the or-vacuous bba $\mathbf{1}_\varnothing$. We first deduce the $\mathbf{K}_X$ matrices for the case $|\Omega| = 1$ and $|\Omega| = 2$, and proceed with the general case.

4.1. Case $\Omega = \{\Omega\}$.

Suppose Ω has only one element. We have $\mathbf{K}_\varnothing = \mathbf{I}$. As $\mathbf{K}_\Omega$ is stochastic, there are $\alpha,\beta \in [0,1]$ such that:

$$\mathbf{K}_\Omega = \begin{bmatrix} \beta & 1-\alpha \\ 1-\beta & \alpha \end{bmatrix}.$$

By P2, $\begin{bmatrix} \beta \\ 1-\beta \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, so $\beta = 0$, and

$$\mathbf{K}_\Omega = \begin{bmatrix} 0 & 1-\alpha \\ 1 & \alpha \end{bmatrix}.$$

These two matrices satisfy all the required properties.

4.2. Case $\Omega = \{a,b\}$.

We look now to the case $\Omega = \{a,b\}$. We have $\mathbf{K}_\varnothing = \mathbf{I}$. Let

$$\mathbf{K}_a = \begin{bmatrix} 0 & 1-\alpha & x & p \\ 1 & \alpha & y & q \\ 0 & 0 & z & r \\ 0 & 0 & t & s \end{bmatrix}$$

The block of 0 results from P6. Let P be the permutation P(a) = b and P(b) = a. Then $\mathbf{L}_P$ is given in section 2.3 and $\mathbf{L}_P^{-1} = \mathbf{L}_P$.
With $\mathbf{K}_b = \mathbf{L}_P^{-1}\,\mathbf{K}_a\,\mathbf{L}_P$ (P4) and $\mathbf{K}_a\,\mathbf{K}_b = \mathbf{K}_b\mathbf{K}_a$ (P5), one has:

$$\mathbf{K}_a\,\mathbf{L}_P^{-1}\,\mathbf{K}_a\,\mathbf{L}_P = \mathbf{L}_P^{-1}\,\mathbf{K}_a\,\mathbf{L}_P\,\mathbf{K}_a\,.$$

So $(\mathbf{K}_a\,\mathbf{L}_P^{-1}\,\mathbf{K}_a\,) = \mathbf{L}_P^{-1}\,(\mathbf{K}_a\,\mathbf{L}_P\,\mathbf{K}_a)\,\mathbf{L}_P^{-1}$.

A (very) tedious analysis of the last equality leads to the solution to x = y = z = 0, t = 1, p = q = 0, s = α. So we obtain unique solutions for $\mathbf{K}_a$ and $\mathbf{K}_b$.

$$\mathbf{K}_a = \begin{bmatrix} 0 & 1-\alpha & 0 & 0 \\ 1 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 1-\alpha \\ 0 & 0 & 1 & \alpha \end{bmatrix} \qquad \mathbf{K}_b = \begin{bmatrix} 0 & 0 & 1-\alpha & 0 \\ 0 & 0 & 0 & 1-\alpha \\ 1 & 0 & \alpha & 0 \\ 0 & 1 & 0 & \alpha \end{bmatrix}$$

Consider $\mathbf{K}(\mathbf{K}_a\mathbf{1}_b) = \mathbf{K}_a\mathbf{K}_b$ (P5). We have $\mathbf{K}_a\mathbf{1}_b = \mathbf{1}_\Omega$, so $\mathbf{K}(\mathbf{K}_a\mathbf{1}_b) = \mathbf{K}(\mathbf{1}_\Omega) = \mathbf{K}_\Omega$. Hence

$$\mathbf{K}_\Omega = \mathbf{K}_a\mathbf{K}_b = \begin{bmatrix} 0 & 0 & 0 & (1-\alpha)^2 \\ 0 & 0 & 1-\alpha & \alpha(1-\alpha) \\ 0 & 1-\alpha & 0 & \alpha(1-\alpha) \\ 1 & \alpha & \alpha & \alpha^2 \end{bmatrix}.$$

We have thus obtained all the needed matrices, $\mathbf{K}(\mathbf{m})$ is fully defined... and depends only on one parameter α which varies on [0,1]. In particular, when $\alpha = 0$, $\mathbf{K}(\mathbf{m}) = \mathbf{F}(\mathbf{m})$, and when $\alpha = 1$, $\mathbf{K}(\mathbf{m}) = \mathbf{E}(\mathbf{m})$ (see section 2). So $\alpha = 0$ corresponds to the exclusive disjunction and $\alpha = 1$ to the disjunction. All other values of α in [0,1] correspond to new disjunctive combination operators.

4.3. The canonical decomposition of K(m).

It is worth looking at the canonical decomposition of $\mathbf{K}(\mathbf{m})$ into its eigenvalues - eigenvectors structure when $\Omega = \{a,b\}$, as a nice structure will emerge. Let

$$\Lambda_{\emptyset} = I,\ \Lambda_a = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\alpha \end{bmatrix},\ \Lambda_b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\alpha & 0 \\ 0 & 0 & 0 & -\alpha \end{bmatrix},\ \Lambda_\Omega = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 \\ 0 & 0 & -\alpha & 0 \\ 0 & 0 & 0 & \alpha^2 \end{bmatrix}.$$

Let $\mathbf{G} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -\alpha & 1 & -\alpha \\ 1 & 1 & -\alpha & -\alpha \\ 1 & -\alpha & -\alpha & \alpha^2 \end{bmatrix}$.

Then $\mathbf{G}^{-1}\,\Lambda_X\,\mathbf{G} = \mathbf{K}_X$ for all $X \subseteq \Omega$.

It happens that all $\mathbf{K}_X$ for $X \subseteq \Omega$ share the same left and right eigenvectors. This decomposition allows to derive a nice representation of $\mathbf{m}_{12} = \mathbf{K}(\mathbf{m}_1)\mathbf{m}_2$.

Lemma 2: Let $\mathbf{g}_1 = \mathbf{Gm}_1$, $\mathbf{g}_2 = \mathbf{Gm}_2$, $\mathbf{g}_{12} = \mathbf{Gm}_{12}$. Then :

$$g_{12}(X) = g_1(X)\, g_2(X) \qquad \text{for all } X \subseteq \Omega.$$

Proof: We have:

$$\mathbf{m}_{12} = \sum_{X \subseteq \Omega} m_1(X)\ \mathbf{K}_X\ \mathbf{m}_2$$

$$= \sum_{X \subseteq \Omega} m_1(X)\ \mathbf{G}^{-1}\ \Lambda_X\ \mathbf{G}\ \mathbf{m}_2$$

and $$\mathbf{Gm}_{12} = \sum_{X \subseteq \Omega} m_1(X)\ \Lambda_X\ \mathbf{G}\ \mathbf{m}_2.$$

With $\mathbf{g}_1 = \mathbf{Gm}_1$, $\mathbf{g}_2 = \mathbf{Gm}_2$, $\mathbf{g}_{12} = \mathbf{Gm}_{12}$,

so $$\mathbf{g}_{12} = \sum_{X \subseteq \Omega} m_1(X)\ \Lambda_X\ \mathbf{g}_2.$$

The relation between $\mathbf{G}$ and Λ_X is such that

$$\sum_{X \subseteq \Omega} m_1(X)\ \Lambda_X = [\text{diag}\ \mathbf{g}_1]$$

Thus $g_{12}(X) = g_1(X)\, g_2(X)$ for all $X \subseteq \Omega$. QED

The lemma 2 relation is nothing but the analogous of the relation for Dempster's rule of combination when g is the commonality function. So $\mathbf{g} = \mathbf{Gm}$ is the analogous of the communality function within the generalized context of the α-disjunction.

The vector $\mathbf{g} = \mathbf{Gm}$ is nothing but the vector of eigenvalues of the matrix $\mathbf{K}(\mathbf{m})$, and $\mathbf{G}$ is a matrix which lines are the left-eigenvectors of $\mathbf{K}(\mathbf{m})$.

4.4. Extending the results to any Ω.

The generalization to any Ω is obtained by iteration. The next theorem describes the strucutre of $\mathbf{K}(\mathbf{m})$.

Theorem 11: For any Ω, the $\mathbf{K}_X$ matrices are:

1) for $X = \emptyset$, $\mathbf{K}_\emptyset = \mathbf{I}$, $\Lambda_\emptyset = \mathbf{I}$,

2) for $x \in \Omega$ and $\mathbf{K}_{\{x\}} = [\, k^x_{AB}]$, we have:

if $x \notin B$,	$k^x_{AB} =$	1	if $A = B \cup \{x\}$
			0 otherwise
if $x \in B$,	$k^x_{AB} =$	α	if $B = A$
		$(1-\alpha)$	if $x \notin A$, $B = A \cup \{x\}$
		0	otherwise,

and the diagonal elements of the $\Lambda_{\{x\}}$ matrices are:

if $x \notin B$	$\lambda_x(B,B) =$	1
if $x \in B$	$\lambda_x(B,B) =$	$\alpha - 1$.

3) and for any $\emptyset \neq X \subseteq \Omega$: $\mathbf{K}_X = \prod_{x \in X} \mathbf{K}_{\{x\}}$, $\Lambda_X = \prod_{x \in X} \Lambda_{\{x\}}$.

4) The X'th column of $\mathbf{G}$ is $\Lambda_X \mathbf{1}$.

Proof: Obtained by iteration. Let $\Omega = \{a,b,c\}$. We have $\mathbf{K}_\emptyset = \mathbf{I}$. By considering bba focused on $\{a,b\}$, one obtains $k^a_{XY} = 0$ for $X \not\subseteq \{a,b\}$, $Y \subseteq \{a,b\}$. Then using bba focused on $\{a,c\}$, one gets the values of k^a_{XY} for $Y = \{c\}$ and $\{a,c\}$. The values of k^a_{XY} for $Y = \{b,c\}$ and $\{a,b,c\}$ are derived by as very tedious computation as in section 4.2. The values of $\mathbf{K}_{\{b\}}$ and $\mathbf{K}_{\{c\}}$ are derived through the application of $\mathbf{L}_P$ matrices. Finally the property P5 allows the derivation of:

$$\mathbf{K}_X = \prod_{x \in X} \mathbf{K}_{\{x\}} \qquad \text{for all } X \subseteq \Omega.$$

Going from spaces Ω with three to four elements and more is performed identically. QED

The fact that the combination can be achieved by pointwise multiplications as with the commonality functions is shown in the nest theorem. This property is very usefull as the Fast Möbius Transform could be adapted to compute g and m from each other. Then the computation of the combination is obtained by transforming each bba into its corresponding g vector, combining the g vectors by pointwise multiplications, and transforming back the result into a bba (Kennes and Smets, 1990, Kennes, 1992).

Theorem 12: Let $\mathbf{g}_1 = \mathbf{G}\mathbf{m}_1$, $\mathbf{g}_2 = \mathbf{G}\mathbf{m}_2$, $\mathbf{g}_{12} = \mathbf{G}\mathbf{m}_{12}$,
then $g_{12}(A) = g_1(A) g_2(A)$ for all $A \subseteq \Omega$.

Proof: We still have the property that $\mathbf{K}_X = \mathbf{G}^{-1} \Lambda_X \mathbf{G}$, and proof proceeds as in lemma 1.
QED

5. The α-conjunctive combination.

We consider now that the vacuous bba is the and-vacuous bba $\mathbf{1}_\Omega$.

In order to distinguish between the conjunctive and the disjunctive families of α-junctions, we introduce the notation $\mathbf{K}^{\vee\alpha}(\mathbf{m})$ for what we had derived in the previous section. We define $\mathbf{K}^{\wedge\alpha}(\mathbf{m})$ as the operator dual to $\mathbf{K}^{\vee\alpha}(\mathbf{m})$ that we would have obtained if we had started with the and-vacuous belief function.

The same derivation as for the disjunctive case can be repeated using $\mathbf{K}_\Omega = \mathbf{I}$, instead of $\mathbf{K}_\varnothing = \mathbf{I}$. All results happen to be similar.

Theorem 13: For any Ω, the $\mathbf{K}^{\wedge\alpha}{}_X$ matrices are:

1) for $X = \Omega$, $\mathbf{K}^{\wedge\alpha}{}_\Omega = \mathbf{I}$, $\Lambda^{\wedge\alpha}{}_\Omega = \mathbf{I}$,

2) for $x \in \Omega$ and $\mathbf{K}_{\{x\}} = [\,k^x_{AB}]$, we have:

$$\text{if } x\in B,\quad k^{\bar{x}}_{AB}{}^{\wedge\alpha} = \begin{cases} 1 & \text{if } x\notin A,\ B=A\cup\{x\} \\ 0 & \text{otherwise} \end{cases}$$

$$\text{if } x\notin B,\quad k^{\bar{x}}_{AB}{}^{\wedge\alpha} = \begin{cases} \alpha & \text{if } A=B \\ (1\text{-}\alpha) & \text{if } A=B\cup\{x\} \\ 0 & \text{otherwise,} \end{cases}$$

and the diagonal elements of the $\Lambda_{\{x\}}$ matrices are:

$$\text{if } x\notin B \qquad \lambda^{\wedge\alpha}{}_{\bar{x}}(B,B) = \ 1$$

$$\text{if } x\in B \qquad \lambda^{\wedge\alpha}{}_{\bar{x}}(B,B) = \ \alpha-1.$$

3) and for any $\Omega\neq X\subseteq\Omega$: $\quad \mathbf{K}^{\wedge\alpha}{}_X = \prod_{x\notin X} \mathbf{K}^{\wedge\alpha}{}_{\{\bar{x}\}}, \quad \Lambda^{\wedge\alpha}{}_X = \prod_{x\notin X} \Lambda^{\wedge\alpha}{}_{\{\bar{x}\}}.$

4) The X'th column of $\mathbf{G}^{\wedge\alpha}$ is $\Lambda^{\wedge\alpha}{}_X\mathbf{1}$.

The links between $\mathbf{K}^{\wedge\alpha}$ and $\mathbf{K}^{\vee\alpha}$ are shown in the next theorem, that is at the core of their De Morgan nature.

Theorem 14: $\qquad \mathbf{K}^{\wedge\alpha}(\mathbf{m}) = \mathbf{J}.\mathbf{K}^{\vee\alpha}(\mathbf{J}.\mathbf{m}).\mathbf{J}.$

or equivalently: for any $A,X,Y\subseteq\Omega$, $k^{X}_{A\,Y}{}^{\wedge\alpha} = k^{\overline{X}}_{\overline{A}\,\overline{Y}}{}^{\vee\alpha}$.

This relation leads to the analogous of the De Morgan formula extended to α-junctions. We use the obvious notations:

$\mathbf{m}[E_1]\wedge_\alpha\mathbf{m}[E_2]$	for	$\mathbf{K}^{\wedge\alpha}(\mathbf{m}[E_1])\mathbf{m}[E_2]$, and
$\mathbf{m}[E_1\wedge_\alpha E_2]$	for	$\mathbf{m}[E_1]\wedge_\alpha\mathbf{m}[E_2]$.

Similarly, we define:

$\mathbf{m}[E_1]\vee_\alpha\mathbf{m}[E_2]$	for	$\mathbf{K}^{\vee\alpha}(\mathbf{m}[E_1])\mathbf{m}[E_2]$, and
$\mathbf{m}[E_1\vee_\alpha E_2]$	for	$\mathbf{m}[E_1]\vee_\alpha\mathbf{m}[E_2]$.

Then with $\mathbf{J}.\mathbf{m}[E] = \mathbf{m}[\sim E]$, we have:

$$\mathbf{K}^{\wedge\alpha}(\mathbf{m}[E_1])\mathbf{m}[E_2] = \mathbf{J}.\mathbf{K}^{\vee\alpha}(\mathbf{J}.\mathbf{m}[E_1])\mathbf{J}.\mathbf{m}[E_2].$$

It becomes:

$\mathbf{m}[E_1]\wedge_\alpha\mathbf{m}[E_2] = \mathbf{m}[E_1\wedge_\alpha E_2] = \mathbf{J}.(\mathbf{m}[\sim E_1]\vee_\alpha\mathbf{m}[\sim E_2])$

$\qquad = \mathbf{J}.\mathbf{m}[\sim E_1\vee_\alpha\sim E_2] = \mathbf{m}[\sim(\sim E_1\vee_\alpha\sim E_2)]$.

So the bba induced by $E_1\wedge_\alpha E_2$ and $\sim(\sim E_1\vee_\alpha\sim E_2)$ are identical, what translates that $E_1\wedge_\alpha E_2$ and $\sim(\sim E_1\vee_\alpha\sim E_2)$ are equal, what is the De Morgan property.

In particular, when $\alpha = 1$, $\mathbf{K}^{\vee 1}(\mathbf{m})$ is the disjunctive operator and $\mathbf{K}^{\wedge 1}(\mathbf{m})$ is the conjunctive operator. The bba $\mathbf{m}_1\wedge_1\mathbf{m}_2$ is the one obtained by applying the conjunctive rule of combination (Dempster's rule of combination unnormalized) to $\mathbf{m}_1$ and $\mathbf{m}_2$.

Deriving $\alpha = 1$.

How to derive the conjunctive and disjunctive rules of combination (hence the $\mathbf{K}^{\wedge 1}$ and $\mathbf{K}^{\vee 1}$ operators)? Thus how to justify $\alpha = 1$? It happens that the only α-junction operator that acts as a specialization (generalization) is obtained with the 1-conjunction (1-disjunction) operator (Klawonn and Smets, 1992). So requiring that $\mathbf{K}(\mathbf{m})$ acts as a specialization (generalization) on any bba implies that $\alpha = 1$, thus leads to the conjunctive rule of combination (and its disjunctive counterpart, the disjunctive rule of combination).

The case $\alpha = 0$.

Suppose two pieces of evidence E1 and E2 and their induced bba m and m. We mentioned in section 1 that:

1) the 1-conjunction ($\mathbf{K}^{\wedge 1}$) correponds to the case where You accept that both sources tell the truth,

2) the 1-disjunction ($\mathbf{K}^{\vee 1}$) correponds to the case where You accept that at least one source tells the truth,

3) the 0-disjunction ($\mathbf{K}^{\vee 0}$) correponds to the case where You accept that one and only one source tells the truth, and You don’t know which is which.

The $\mathbf{K}^{\wedge 0}$ operator does not have a name: it fits with the case where You know that either none of or both sources tell the truth, a quite artifical case in practice.

The practical interest of the $\alpha = 0$ cases are limited. This might explain why they were not introduced previously. In any case, $\alpha = 0$ should not be understood as intermediate between the 1-conjunctive and 1-disjunctive rules. In fact, the $\mathbf{K}^{\vee\alpha}$ operator is intermediate between the $\mathbf{K}^{\vee 1}$ and the $\mathbf{K}^{\vee 0}$ operators, and the $\mathbf{K}^{\wedge\alpha}$ operator is intermediate between the $\mathbf{K}^{\wedge 1}$ and the $\mathbf{K}^{\wedge 0}$ operators.

Remark: In set theory, two operators, the joint denial and Sheffer’s stroke, can be used to represent the AND, the OR and the negation with a unique symbol. We cannot extend this result to the α-junctions. Indeed their definition bears strongly on the idempotency property and $\mathbf{K}(\mathbf{m})$ usually does not satisfy $\mathbf{K}(\mathbf{m})\mathbf{m} \neq \mathbf{m}$. Hence it seems to be hopeless to find the analogous of these two special operators.

6. Some comments.

6.1. Explaining the negation ~E.

Suppose the bba $\mathbf{1}_\Omega$ defined on Ω. Then $\mathbf{K}^{\vee 0}(\mathbf{1}_\Omega) = \mathbf{J}$. So $\mathbf{K}^{\vee 0}(\mathbf{1}_\Omega)\mathbf{m}[E] = \mathbf{J}.\mathbf{m}[E] = \mathbf{m}[\sim E]$. As $\mathbf{1}_\Omega$ can be seen as the bba induced by the piece of evidence that supports nothing specific on Ω, we can define $\mathbf{1}_\Omega$ as $\mathbf{m}[T]$ where T denotes the vacous piece of evidence, i.e., the triple (S, T, true) where T is a tautology. In particular, $E\vee T = T$ and $E\wedge T = E$ for any E. Therefore $\mathbf{K}^{\vee 0}(\mathbf{1}_\Omega)\mathbf{m}[E] = \mathbf{K}^{\vee 0}(\mathbf{m}[T])\mathbf{m}[E] = \mathbf{m}[\sim E]$. So we obtain an explanation of the meaning of $\mathbf{m}[\sim E]$ as being the bba induced by an exclusive disjunction between E and T: $\sim E = E \underline{\vee} T$.

In practice, ~T is impossible (hopefully, if one hopes to develop a realistic model: the souce states a tautology and if ~T holds, it means You accept that the source tells the false). So when $E\underline{\vee}T$ holds, it means You accept that the source that states E is telling the false. So ~E represents the bba that would be induced if You know that the source is telling the false: whenever the source give a support that the actual world ω_0 belongs to A, You give that support to $\overline{A}$.

6.2. Spread of $m_1(X)m_2(Y)$ on Ω.

The relations $\mathbf{K}^{\vee\alpha}(\mathbf{m}_1)\mathbf{m}_2$ and $\mathbf{K}^{\wedge\alpha}(\mathbf{m}_1)\mathbf{m}_2$ can also be represented in such a way that one realizes that both combination operators correspond to a distribution of the product $m_1(X)m_2(Y)$ among some specific subsets of Ω. In fact the Y'th column of $\mathbf{K}^{\vee\alpha}{}_X$ and $\mathbf{K}^{\wedge\alpha}{}_X$ is the probability distribution according to which the mass $m_1(X)m_2(Y)$ is distributed on the subsets of Ω. So the terms $k^X_{A\,Y}{}^{\vee\alpha}$ and $k^X_{A\,Y}{}^{\wedge\alpha}$ are the proportions of $m_1(X)m_2(Y)$ that is allocated to A after the $\vee_\alpha$ and $\wedge_\alpha$ combinations of $\mathbf{m}_1$ and $\mathbf{m}_2$, respectively. The symmetry of the product $m_1(X)m_2(Y)$ is translated by the fact that both $k^X_{A\,Y}{}^{\vee\alpha}$ and $k^X_{A\,Y}{}^{\wedge\alpha}$ are both symmetric in X and Y.

6.3. Measure of the impact of K(m).

A natural measure of the impact of the operator $\mathbf{K}(\mathbf{m})$ is its determinant $|\mathbf{K}(\mathbf{m})|$. It happens that $|\mathbf{K}(\mathbf{m})| = \prod_{X\subseteq\Omega} g(X)$ where the g(X) terms are the eigenvalues of $\mathbf{K}(\mathbf{m})$ (see section 4.3). This relation was already obtained for the 1- conjunction (Smets, 1983) where we understood the product as a measure of the information contains in $\mathbf{m}$. We think this was inappropriate and the idea of 'impact' is better.

7. Conclusions.

In conclusion, we have discovered a family of α–junction operators that include as particular cases the conjunctive rule of combination, the disjunctive rule of combination, the exclusive disjunctive rule of combination, and their negations. The operators $\mathbf{K}^{\vee\alpha}$ (and its dual $\mathbf{K}^{\wedge\alpha}$) generalize the classical concept of conjunction and disjunction within the context of belief function, i.e., a particular context of weighted sets. The requirements that underlie the derivation of the structure of this operator are those expected by a belief function. Their extension to other theories are not obvious. For instance, using our approach for fuzzy sets and possibility theory will probably be inadequate as the linearity requirement is not the kind of requirement assumed within these two theories.

The meaning of $\mathbf{K}^{\vee\alpha}$ and $\mathbf{K}^{\wedge\alpha}$ is clear with $\alpha = 0$ or 1. With other values of α, their meaning need further study. At least, we have shown that the classical conjunction and disjunction operations are just extreme cases of a general theory and that a continuum of operators can be built between the conjunction and $\underline{\wedge}$, and between the disjunction and the exclisive disjunction.

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We can also define the concept of 'equivalence at a static level'. We can often describe several combinations of sources and propositions that would induce the same state of belief for You. As an example, consider the conjunctive combination $E1 \wedge E2$. Consider another source S* that states $E1 \wedge E2$, and You accept S* tells the truth. In both cases You would be in the same state of belief, as You get the same information. But further information could show the differences between the two cases. With the two sources story, You might receive a new information that states that the source S2 is not telling the truth, in which case You would retract the piece of evidence E2 and keep only E1. So even though You seem to be in a similar state of belief with the two sources and with the one source stories, the two cases are potentially different.

Coming back to the two soureces S1 and S2 where S1 states E1 and S2 states S2, we have:
1. $E1 \wedge E2$ is equivalent at the static level to one source stating $E1 \wedge E2$ and You accept it tells the truth. It is equivalent to 'You accept $E1 \wedge E2$'.
2. $E1 \vee E2$ is not equivalent at the static level to one source stating $E1 \vee E2$ and You accept it tells the truth. This implies that You accept $E1 \vee E2$, but it means more as the reverse implication is not valid.
3. $E1 \underline{\vee} E2$ is equivalent to two source stating E1 and E2, respectively and You accept that at least one is telling the truth and another stating $E1 \wedge E2$, and You accept that the frist is telling the truth and the second is telling the false. So You accept $E1 \underline{\vee} E2$ (the disjunctive disjunction, what is equivalent to $(E1 \vee E2) \wedge \neg(E1 \wedge E2)$)