3D Hybrid Formation Control of an Underwater Robot Swarm: Switching Topologies, Unmeasurable Velocities, and System Constraints

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3D hybrid formation control of an underwater robot swarm: Switching topologies, unmeasurable velocities, and system constraints

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Abstract

This paper addresses formation control of underactuated autonomous underwater vehicles in three-dimensional space, using a hybrid protocol that combines aspects of centralized and decentralized control with constraints that are particular to underwater vehicles, including switching topologies, unmeasurable velocities, and system constraints. Using a distributed leader-follower model, the hybrid formation protocol does not require velocity sensing, access to global information, or static and connected topologies. To handle switching jointly connected networks—that is, to tolerate temporary disconnections—a distributed observer is designed for followers to cooperatively estimate leader states using local measurements and local interactions. On this basis, a compound formation control strategy is proposed to achieve geometric convergence. Firstly, cascaded extended state observers are developed to recover the unmeasurable velocities and unknown dynamic uncertainties induced by internal model uncertainty and external disturbances. Secondly, an improved three-dimensional line-of-sight guidance law at the kinematic level is used to address the underactuated configuration and the nonzero attack and sideslip angles. Thirdly, to overcome potential instability as a result of system constraints, including velocity constraints and input saturations, two adaptive compensators in the dynamic controller are used to address the negative effects of truncation. Using the proposed approach, the estimation errors and formation tracking errors are proved to be uniformly and ultimately bounded. Additionally, the numerical simulation results verify the performance of the approach and demonstrate improvement over both distributed and centralized state-of-the-art approaches.

Keywords: Formation control, Autonomous underwater vehicles, Distributed observer, Switching topologies, System constraints

1. Introduction

In distributed control of robot swarms, both opinion dynamics and physical coordination have been widely studied. Consensus achievement in swarm robotics has, for instance, been studied in flocking problems and in opinion formation in collective decision-making (e.g., [1]). Formation control, by contrast, is more intensively studied in control theory [2], as noted in [3]. Swarm...
robotics has advanced significantly in recent years, especially concerning swarms of small, simple
ground or aerial robots managed primarily by kinematic control (e.g., [4, 5]). However, swarms
of other robot types such as autonomous underwater vehicles (AUVs) require the consideration
of new practical challenges, as do tasks such as formation control that have not frequently been
studied in swarm robotics. These challenges are difficult to address with purely decentralized
control, even from a swarm robotics perspective—instead, they are well-suited to hybrid control
approaches that combine aspects of centralized and decentralized control, and which are recently
becoming more common in swarm robotics research [e.g., with hierarchy 6–8]. Likewise, formation
protocols for AUVs are becoming more widespread, but some important practical challenges of
underwater swarms have not yet been integrated theoretically.

1.1. Motivations

Although formation control of AUVs has received much recent attention (as reported in [9]), the
state of the art still has several important gaps that prevent theoretical studies from being trans-
ferred to applications. The contribution of this paper is theoretical, and addresses the following
three challenges that are of practical importance for formation control of AUV swarms:

(1) **Network reliability:** To achieve the desired behavior cooperatively, AUVs are required to
exchange information using wireless communication. In underwater environments, the most
commonly used technique is acoustic communication, due to its low attenuation [10]. In addi-
tion to intrinsic constraints such as limited bandwidth and range, acoustic communication is
not perfectly reliable. As with many communication techniques, the occurrence of occasional
link failures is inevitable. Additionally, the reliability of acoustic communication can be greatly
affected by environmental conditions such as temperature and salinity [11]. Overall, network
unreliability in underwater conditions means that it is difficult or impossible to maintain a
connected and static communication topology in an AUV swarm in practice. It is therefore of
great importance to study not only graphs with switching topologies, but also graphs that can
have disconnections.

(2) **Position and velocity measurement:** Formation protocols in the literature normally re-
quire both global position and velocity measurements, essentially transforming the problem of
formation control of the swarm into simple single-vehicle trajectory tracking for each AUV sep-
arately. However, in underwater robot swarms there are problems with assuming that both the
global position and velocity measurements will be available. Regarding the former, real global
position information needs to be based on pre-deployed and localized infrastructures, and in
practice it is difficult for AUVs to consistently make use of widely available infrastructure
such as GPS, due to signal attenuation problems [12]. Realistically, underwater robot swarms
should be assumed to operate (at least some of the time) in GPS-denied environments, in
which AUVs need to directly measure their relative positions to their neighbors (e.g., using si-
nultaneous localization and mapping, monocular vision feature matching, or baseline acoustic
positioning [13]). Regarding the latter, if real AUVs can be equipped with velocity sensors at
all (which might be prohibitively too costly and/or heavy), accurate velocity measurements are
still an unresolved challenge due to sensor noise and oceanic disturbances [14]. This motivates
the development of a velocity-free formation protocol using relative position measurements,
which is important for reliability in a challenging environment such as deep water.
Velocity constraints: In order to meet safety and formation tracking performance requirements, a formation protocol must keep the velocities and control inputs of AUVs within certain compact sets. Any violation of these system constraints could result in degraded or even unstable behavior [15]. In the literature, control input constraints have been covered extensively (e.g., [16]), but velocity constraints in AUVs have rarely been discussed. The exceptions are some optimization-based methods (e.g., reference governor [15, 17] or model predictive control [18]) to meet velocity constraints. However, because these approaches solve the optimization problem online, their efficacy is dependent on whether the formulated optimization problem can be solved reliably and quickly enough in real time. In this sense, existing protocols do not necessarily guarantee velocity constraint satisfaction. Therefore, AUV formation protocol with guaranteed system constraint satisfaction, which is crucial for the practical handling of real AUVs, can be considered an unsolved challenge.

To address the challenge of network reliability, this paper studies formation control for AUV swarms that can have switching jointly connected topologies. To address the challenges of position and velocity measurement and velocity constraints, it studies formation control that does not require velocity sensing but still guarantees velocity constraint satisfaction. In other words, instead of being exclusive to velocity sensing, this paper studies formation control that can be combined with any sensing technique that can be used to calculate relative position.

1.2. Related Work

Approaches to the formation control problem are numerous, but few of them are suitable for AUVs. Table 1 summarizes the state of the art in terms of the main contributions of this paper—i.e., summarizes existing approaches in terms of the topology types that were considered, the measurements required for the approach, the constraints for which satisfaction can be guaranteed, and the dimensions handled. For marine vehicles, leader-follower strategies have been widely studied because of their simplicity and scalability, ranging from two-dimensional (2D) plane (e.g. [19–25]) to three-dimensional (3D) space (e.g. [18, 26–31]). In [20], the formation control problem was reformulated as a trajectory tracking problem, where the target trajectory is generated based on leader position and a predefined formation. In a formation learning controller presented in [21], the information of the leader is estimated via a distributed estimator.

Note that the above approaches are applicable only if the communication topology is static. Such an assumption does not hold practically in an underwater environment, and a few approaches have begun to address this. In [32], sufficient conditions were derived to achieve the target formation under both fixed and switching topologies, and in [33, 34], Yan et al. addressed discrete-time formation control for AUVs subject to weak communication constraints, including switching topologies and packet losses. Even so, all possible topologies in [32–34] are supposed to be connected. From a practical point of view, it is difficult to guarantee connectivity at all times, due to unavoidable link failures in an underwater environment. Therefore, it is more relevant to investigate formation control under switching jointly connected topologies—in other words, topologies that might be disconnected at any time.

One key limitation of existing approaches for AUVs is the reliance on mandatory velocity measurements for each vehicle (as shown in Table 1), despite velocities being unmeasurable. Velocity sensors, e.g., Doppler Velocity Log, are active sensing devices with high energy consumption, and are difficult to integrate in small and low-cost vehicles with limited payload capacity. In addition, velocity information cannot be calculated via direct derivative/filtering of global position for the...
Table 1: Differences among existing formation control approaches for AUV swarms, in terms of topology types, required measurements, guaranteed constraints, and dimensions.

<table>
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<th>Topology</th>
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<td>Switching and connected topologies</td>
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following reasons: i) accurate global position measurements should be performed at a relatively high update rate, which is not possible when AUVs are in deep underwater conditions; and ii) even if GPS signals are available near the surface, the numerical differentiation of noisy position measurements leads to chattering of the actuators, and the use of low-pass filters on the numerically reconstructed velocities could deteriorate the tracking performance. An admissible way to recover the unmeasurable states is by using observers [35]. In [25], velocity sensors were replaced by introducing an auxiliary finite-time velocity observer, however only AUVs in a horizontal plane were considered. The velocity observer design in 3D space is more complicated due to highly coupled kinematics and strong nonlinear dynamics. An initial study on this topic is presented in [31], wherein a state-transformation-based velocity observer was designed based on neural networks. However, global position should still be consistently measurable in order to employ the velocity observer proposed by [31]. To the best of the authors’ knowledge, velocity-free formation control of AUV swarms is still an open challenge in 3D space, especially when only relative position measurement is available. This is an important gap to address, as velocities are often unmeasurable and there are many alternatives (i.e., velocity-free GPS-denied positioning techniques) available [13].

Another shortcoming of the existing methods is that they cannot preserve optimal control performance under system constraints comprehensively (i.e., including both velocity constraints and input saturations). As seen in Table 1, input saturations have been investigated for formation control of AUVs, for instance in [22–24, 29, 30], but none of these studies have included velocity constraints. By contrast, in [18], a receding horizon algorithm was designed to achieve a prescribed geometric pattern with satisfaction of velocity saturations, but input constraints were not studied. In [15, 17], reference governors (RGs) were used to optimize command signals within system constraints, to handle both velocity and input saturations. Note that, because both of these are optimization-based methods, a feasible solution must be newly found in each control cycle. Therefore, in cases where it is non-trivial to solve the formulated optimization problem in real time, constraint satisfaction is not in this sense guaranteed. In general, the constrained formation control of underactuated AUVs in 3D space is currently an unsolved challenge.
1.3. Contributions

Motivated by the current gaps in the state of the art, this research proposes a hybrid formation protocol for underactuated AUVs in 3D space. The main contributions are summarized as follows:

(1) The proposed formation protocol is verified to be effective under switching jointly connected topologies (i.e., topologies that might be disconnected at any time), thus relaxing the assumptions in the existing literature of static topologies [18–31] or switching but connected topologies [32–34]. This is realized by the proposed distributed observer (DO), through which each follower estimates the leader’s information, including relative position, orientation, and velocity.

(2) The proposed formation protocol removes the limitation of reliance on velocity sensing, as compared to [18–24, 26–30]. The unmeasurable velocities, as well as the dynamic uncertainties, are recovered by the proposed cascaded extended state observer (CESO), which is constructed by state and coordinate transformations to satisfy a standard integral-chain form.

(3) In addition to input saturation as covered in [22–25, 29, 30], the proposed approach simultaneously guarantees velocity constraint satisfaction, without relying on optimization-based methods [15, 17, 18] that are not guaranteed to be effective due to the need of finding a solution in every control cycle. To this end, two adaptive wind-up compensators are introduced to alleviate the negative effect of nonlinear saturations on system stability. In addition to guaranteeing constraint satisfaction, the proposed approach is simpler to implement than optimization-based methods.

1.4. Paper organization

The paper is organized as follows. First, section 2 introduces the preliminaries, including graph theory and underactuated AUV modeling, and formulates the formation control problem. Then the proposed hybrid formation protocol is introduced in two parts: the design and analysis of the DO (Sec. 3) and compound velocity-free formation control (Sec. 4). Finally, the closed-loop system stability under the proposed method is analyzed (Sec. 5), the comparative simulation results are presented (Sec. 6), and the paper is concluded.

Notation: $\mathbb{R}^{n \times m}$ denotes the set of $n \times m$ real matrices. $|\cdot|$ and $\|\cdot\|$ denote the $L_1$-norm and $L_2$-norm. $\lambda_{\min}(\cdot)$ is the smallest eigenvalue of a square matrix. $\mathbf{R}_x^{A}(\alpha, \beta) = \mathbf{R}_x(\alpha) \mathbf{R}_y(\beta)$ is the rotation matrix from frame $\{A\}$ to frame $\{B\}$ with $\mathbf{R}_x(\alpha) = [\cos \alpha, -\sin \alpha, 0; \sin \alpha, \cos \alpha, 0; 0, 0, 1]$ and $\mathbf{R}_y(\beta) = [\cos \beta, 0, \sin \beta; 0, 1, 0; -\sin \beta, 0, \cos \beta]$. $\text{sgn}(\cdot)$ is the standard sign function, and $\text{sig}^\alpha(x) = \text{sgn}(x)|x|^\alpha$, where $\alpha \geq 0$.

2. Preliminaries and Problem Formulation

2.1. Graph theory

In leader-follower formation control, the interactions among $N$ followers can be modeled by graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, in which $\mathcal{V} = \{1, \ldots, N\}$ is a finite set of nodes, $\mathcal{E} = \{(i, j) : i, j \in \mathcal{V}, i \neq j\}$ is a set of edges. Assume that graph $\mathcal{G}$ is undirected. The adjacency matrix of graph $\mathcal{G}$ is denoted by $\mathbf{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$ with nonnegative weights. If $(i, j) \in \mathcal{E}$, then the bidirectional exchange of information between follower $i$ and follower $j$ corresponds to weight $a_{ij} = 1$. By contrast, if $(i, j) \notin \mathcal{E}$, then $a_{ij} = 0$. Note that $a_{ii} = 0, \forall i \in \mathcal{V}$. The neighbor set of follower $i$ is defined as
The Laplacian matrix $\mathbf{L} = [l_{ij}]_{N \times N}$ associated with graph $\mathcal{G}$ is defined as follows: $l_{ij} = -a_{ij}$ if $i \neq j$, and $l_{ij} = \sum_{j=1}^{N} a_{ij}$ if $i = j$. Communication links between the leader and followers can be characterized by adjacency matrix $\mathbf{B} = \text{diag} \{ b_1, \ldots, b_N \}$, where $b_i = 1$ if follower $i$ connects to the leader. Let the leader be represented by node 0, and derive the augmented graph $\bar{G} = (\bar{V}, \bar{E})$, where $\bar{V} = V \cup \{0\}$ and $\bar{E}$ includes $E$ and also includes edges between leader and followers. Denote matrix $\mathbf{H} \in \mathbb{R}^{N \times N}$ as $\mathbf{H} = \mathbf{L} + \mathbf{B}$.

The communication links of the AUVs are assumed to be dynamically switching in practice, due to unreliable acoustic communication and constrained communication range. Denote all possible graphs of multiple AUVs as $\mathcal{S} = \{ \mathcal{G}_q : q \in \mathcal{Q} \}$, where $\mathcal{Q}$ is an index set of $\mathcal{S}$. Consider an infinite sequence of non-overlapping bounded time intervals $[t_k, t_{k+1})$ with $k = 0, 1, \ldots$ and $t_0 = 0$. In the interval $[t_k, t_{k+1})$, there exists a finite time sequence $t_k^0, t_k^1, \ldots, t_k^{l_k-1}$, where $t_0^k = t_k$ and $t_k^{l_k} = t_{k+1}$ for some integer $l_k > 0$. The graph switches at time $t_k^{l_k}$, $j = 0, 1, \ldots, l_k - 1$ and is time-invariant in the subinterval $[t_k^j, t_k^{j+1})$. Assume that there exists a constant number $\tau > 0$ called dwell time, such that $t_k^{j+1} - t_k^j > \tau$ for all time intervals. Denote $\sigma(t) : [0, +\infty) \rightarrow \mathcal{Q}$ as the switching signal. The undirected graph $\bar{G}$ and matrix $\mathbf{H}$ at $\sigma(t)$ are therefore noted respectively as $\bar{G}_{\sigma(t)}$ and matrix $\mathbf{H}_{\sigma(t)}$. Note that $\bar{G}_{\sigma(t)}$ is allowed to be disconnected. A collection of graphs across the time interval $[t, t + T]$ with $T > 0$ is said to be jointly connected if the graph union $\{ \bar{G}_{\sigma(s)} : s \in [t, t + T] \}$ is connected. For each $q \in \mathcal{Q}$, $\mathbf{H}_q$ has $N$ eigenvalues denoted as $\lambda_q^1, \lambda_q^2, \ldots, \lambda_q^N$ based on the labeling rule given in [36]. Define $\mathcal{C}(q) = \{ k : \lambda_q^k \neq 0, k = 1, 2, \ldots, N \}$ and note the following lemma:

**Lemma 1 [36].** The graphs are jointly connected across each interval $[t_k, t_{k+1})$, if and only if $\bigcup_{t \in [t_k, t_{k+1})} \mathcal{C}(\sigma(t)) = \{1, \ldots, N\}$.

### 2.2. Model of underactuated AUV

For formation control of AUVs, kinematics in 3D space is typically described by a 6-DOF model in the directions of surge, sway, heave, roll, pitch, and yaw. Torpedo-shaped AUVs, which are widely used in practice, are typically passively stabilized in the roll direction, and in AUV models it is commonly assumed that the roll angle and rate are zero [37]. Roll motion is therefore excluded in the AUV model.
Allow the subscript $i$ to represent the variables associated with the $i$th follower, for the remainder of this paper. To describe the AUV’s motion, three reference frames are employed (see Fig. 1): earth-fixed frame $\{I\}$, body-fixed frame $\{B_i\}$ and resultant velocity frame $\{W_i\}$. For $i = 1, \ldots, N$, assume the origins of $\{B_i\}$ and $\{W_i\}$ coincides with the center of gravity of the AUV $M_i$. The 5-DOF kinematic and dynamic model described in [38] is used for the $i$th AUV:

$$
\begin{align*}
\dot{\mathbf{q}}_i &= T(\theta_i, \psi_i) \mathbf{v}_i \\
M_i \ddot{\mathbf{v}}_i + C_i(\mathbf{v}_i) \mathbf{v}_i + D_i(\mathbf{v}_i) \mathbf{v}_i + g_i = \tau_i + \tau_{di},
\end{align*}
$$

with $T(\theta_i, \psi_i)$ defined as

$$
T(\theta_i, \psi_i) = \begin{bmatrix}
\cos \theta_i \cos \psi_i & -\sin \psi_i & \sin \theta_i \cos \psi_i & 0 & 0 \\
\cos \theta_i \sin \psi_i & \cos \psi_i & \sin \theta_i \sin \psi_i & 0 & 0 \\
-\sin \theta_i & 0 & \cos \theta_i & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1/\cos \theta_i
\end{bmatrix},
$$

where $\mathbf{q}_i = [x_i, y_i, z_i, \theta_i, \psi_i]^T$, with $x_i, y_i, z_i$ being the coordinates of $M_i$ in frame $\{I\}$, and $\theta_i$ and $\psi_i$ being the pitch and yaw angles. $\mathbf{v}_i = [u_i, v_i, w_i, q_i, r_i]^T$ denotes the velocity vector on surge, sway, heave, pitch, and yaw directions expressed in frame $\{B_i\}$. $M_i \in \mathbb{R}^{5 \times 5}$ is the inertia matrix with added mass, $C_i \in \mathbb{R}^{5 \times 5}$ is the centripetal and Coriolis matrix, $D_i \in \mathbb{R}^{5 \times 5}$ is the damping matrix, $g_i \in \mathbb{R}^5$ denotes the restoring forces and moments due to gravity and buoyancy, $\tau_i \in \mathbb{R}^5$ is the control input vector, $\tau_{di} = [\tau_{dai}, \tau_{dvi}, \tau_{dri}, \tau_{dq}, \tau_{dqi}, \tau_{dr}]^T$ denotes the dynamic uncertainties, including unmodeled hydrodynamics and external disturbances.

Concerning the controllability and observability of AUVs, the following assumptions are made, based on [38].

**Assumption 1:** (a) The torpedo-shaped AUVs have port/starboard and bottom/top symmetries so that $M_i$ and $D_i$ are diagonal and $C_i$ is skew-symmetric. (b) The center of gravity and the center of buoyancy are located vertically on the $z_{B_i}$ axis. (c) $|\theta_i| \neq \pi/2$.

Under Assumption 1, the dynamic model can be reformulated in the following differential form

$$
\begin{align*}
\dot{u}_i &= \frac{m_{ui} u_i v_i - m_{wi} u_i q_i - d_{ui} u_i}{m_{ui}} + \frac{\text{sat}(\tau_{ui})}{m_{ui}} + \frac{\tau_{dai}}{m_{ui}} = \frac{\text{sat}(\tau_{ui})}{m_{ui}} + D_{ui} \\
\dot{v}_i &= -\frac{m_{ui} u_i r_i - d_{ui} v_i}{m_{vi}} = \frac{\tau_{dvi}}{m_{vi}} = D_{vi} \\
\dot{w}_i &= \frac{m_{ui} u_i q_i - d_{ui} w_i}{m_{wi}} + \frac{\tau_{dwi}}{m_{wi}} = D_{wi} \\
\dot{q}_i &= \frac{(m_{ui} - m_{wi}) u_i v_i - d_{q1} q_i - G_i \nabla_i \sin \theta_i}{m_{qi}} + \frac{\text{sat}(\tau_{qi})}{m_{qi}} + \frac{\tau_{dq}}{m_{qi}} = \frac{\text{sat}(\tau_{qi})}{m_{qi}} + D_{qi} \\
\dot{r}_i &= \frac{(m_{ui} - m_{wi}) u_i v_i - d_{r1} r_i}{m_{ri}} + \frac{\text{sat}(\tau_{ri})}{m_{ri}} + \frac{\tau_{dri}}{m_{ri}} = \frac{\text{sat}(\tau_{ri})}{m_{ri}} + D_{ri}
\end{align*}
$$

where $m_{(\cdot)i}$ and $d_{(\cdot)i}$ are inertia and hydrodynamic damping coefficients, respectively; $G_i$ is submerged weight; $\nabla_i$ is the distance between the center of gravity and the center of buoyancy; $D_{(\cdot)i}$ denotes the lumped uncertainties including unknown hydrodynamics and environmental disturbances; and $\text{sat}(\cdot)$ is the input saturation function to be defined later.

Given the inherent operation ranges of the propeller and control surfaces of AUVs, the control inputs $\tau_{(\cdot)i}$ are naturally bounded [29]. The velocity $\mathbf{v}_i$ is also expected to remain within certain
prescribed compact sets for performance and safety considerations. Therefore, the saturation nonlinearity is given by

\[
sat(x_i) = \begin{cases} 
  x_i^{\text{max}}, & x_i > x_i^{\text{max}} \\
  x_i, & x_i^{\text{min}} \leq x_i \leq x_i^{\text{max}} \\
  x_i^{\text{min}}, & x_i < x_i^{\text{min}} 
\end{cases}, \quad x = u, q, r, \tau_u, \tau_q, \tau_r, \tag{4}
\]

where \(x_i^{\text{max}}\) and \(x_i^{\text{min}}\) are the upper and lower bounds, respectively. Note that there are no control inputs in the sway and heave directions. It is commonly assumed that \(v_i\) and \(w_i\) are passively bounded, and therefore will not be considered in the controller design \([2, 19]\).

From Eq. (3), it can be observed that AUVs suffer from variations in the sway and heave directions. Therefore, the resultant velocity direction of an AUV is not parallel to the surge direction, which causes nonzero and time-varying attack angle \(\alpha_i = \arctan(w_i/u_i)\) and sideslip angle \(\beta_i = \arctan(v_i/\sqrt{u_i^2 + w_i^2})\). Define \(\nu_{W_i} = [U_i, 0, 0]^T\), with \(U_i = \sqrt{u_i^2 + v_i^2 + w_i^2}\), \(\nu_{B_i} = [u_i, v_i, w_i]^T\), which represents the resultant velocity of an AUV in frames \(\{W_i\}\) and \(\{B_i\}\). Then, the following transformation form exists:

\[
\nu_{W_i} = \mathbf{R}_{B_i}^{W_i} (-\beta_i, \alpha_i) \nu_{B_i}. \tag{5}
\]

2.3. Leader-follower formation tracking error dynamics

The leader-follower formation control problem is illustrated in Fig. 2, wherein a group of AUVs is required to achieve a desired formation with respect to the leader \(M_0\). Note that the motion control of the leader is not considered in this paper, and the leader is assumed to move on a predefined trajectory. Denote the position of \(M_0\) in \(\{I\}\) frame as \(p_0(t) = [x_0(t), y_0(t), z_0(t)]^T\). Associated with the leader \(M_0\), Serret-Frenet frame \(\{F\}\) is built. Let \(\mathbf{R}_F^I = (\mathbf{R}_F^I (x_0, v_0))^T\) denote the rotation matrix from frame \(\{F\}\) to frame \(\{I\}\), where rotation angles are defined according to \([2]\), as

\[
\begin{aligned}
  \chi_0 &= \arctan2(y_0(t), x_0(t)) \\
  \nu_0 &= \arctan \left( \frac{-\dot{z}_0(t)}{\sqrt{\dot{x}_0^2(t) + \dot{y}_0^2(t)}} \right).
\end{aligned} \tag{6}
\]

Define \(p_i = [x_i, y_i, z_i]^T\) as the inertial position of \(M_i\) and \(h_i = [d_{xi}, d_{yi}, d_{zi}]^T\) as the desired relative position between \(M_i\) and \(M_0\). Then, the formation tracking error built in frame \(\{F\}\) is given by

\[
e_{pi} = [e_{xi}, e_{yi}, e_{zi}]^T = \mathbf{R}_F^I (p_i - p_0 - h_i). \tag{7}
\]

**Remark 1:** The formation tracking error in \([39, 40]\) is defined as

\[
e_{pi}(t) = \sum_{j \in \mathcal{N}_i} (p_i(t) - h_i) - (p_j(t) - h_j). \tag{8}
\]

Note that Eq. (8) requires that the neighbor set \(\mathcal{N}_i\) is not empty (i.e., that no vehicle is temporarily disconnected from all other vehicles at any point in time). By contrast, our formation tracking error is directly defined between leader and followers and a distributed observer will be designed later for followers to cooperatively estimate leader information over jointly connected topologies (as shown in Assumption 1), which tolerates temporary isolations of AUVs. Therefore, the assumption of consistently connected graphs present in \([39, 40]\) is relaxed.
Figure 2: Leader-follower formation control problem. The leader is assumed to move along a predefined trajectory, along which the Serret-Frenet frame $\{F\}$ is built. $d_i$ is the desired relative position vector between the $i$-th follower and the leader, and $Q_i$ denotes the target position of the $i$-th follower in the formation. $e_{xi}$, $e_{yi}$, and $e_{zi}$ are formation tracking errors built in frame $\{F\}$. $\psi_{ei}$ and $\theta_{ei}$ are rotation angles from frame $\{F\}$ to frame $\{B_i\}$.

Differentiating Eq. (7) with respect to time yields

$$\dot{e}_{pi} = S^F_I e_{pi} + R^F_I \dot{p}_i - \nu_0,$$

(8)

where $\nu_0 = [U_0, 0, 0]^T$, with $U_0 = \sqrt{x_0^2 + y_0^2 + z_0^2}$. The matrix $S^F_I$ is defined by

$$S^F_I = \begin{bmatrix}
0 & -\dot{\chi}_0 \cos v_0 & -\dot{\nu}_0 \\
-\dot{\chi}_0 \cos v_0 & 0 & -\dot{\chi}_0 \sin v_0 \\
\dot{\nu}_0 & \dot{\chi}_0 \sin v_0 & 0
\end{bmatrix}.$$

(9)

From kinematic model Eq. (1) and coordinate transformation Eq. (5), it can be derived that

$$\dot{p}_i = R^I_B (\psi_i, \theta_i) \nu_{B_i} = R^I_B (\psi_i, \theta_i) \left( R^W_{B_i} (-\beta_i, \alpha_i) \right)^T \nu_{W_i}.$$

(10)

Introducing Eq. (10) in Eq. (8) yields

$$\dot{e}_{pi} = S^F_I e_{pi} + R^F_I R^I_{B_i} \nu_{B_i} - \nu_0$$

$$= S^F_I e_{pi} + R^F_I (\psi_{ei}, \theta_{ei}) \nu_{W_i} - \nu_0 + f_{pi},$$

(11)

where $\psi_{ei}$ and $\theta_{ei}$ are rotation angles from frame $\{F\}$ to frame $\{B_i\}$ and $f_{pi} = [f_{xi}, f_{yi}, f_{zi}]^T = R^F_I R^I_{B_i} (v_{B_i} - v_{W_i})$. Eq. (11) can then be expanded as

$$\begin{cases}
\dot{e}_{xi} = e_{yi} \dot{\chi}_0 \cos \nu_0 - e_{zi} \dot{\nu}_0 + U_i \cos \psi_{ei} \cos \theta_{ei} - U_0 + f_{xi} \\
\dot{e}_{yi} = -e_{xi} \dot{\chi}_0 \cos \nu_0 - e_{zi} \dot{\chi}_0 \sin \nu_0 + U_i \sin \psi_{ei} \cos \theta_{ei} + f_{yi} \\
\dot{e}_{zi} = e_{xi} \dot{\nu}_0 + e_{yi} \dot{\chi}_0 \sin \nu_0 - U_i \sin \theta_{ei} + f_{zi}
\end{cases}$$

(12)

Remark 2: The nonlinearity $f_{pi}$ is induced by the nonzero attack angle $\alpha_i$ and sideslip angle $\beta_i$, because $f_{pi}$ will be equal to zero if $\alpha_i = \beta_i = 0$. From Eq. (12), the effect of attack angle $\alpha_i$ and
sideslip angle $\beta_i$ on the formation tracking performance can clearly be identified. It is therefore
important to fully consider this nonlinearity in the controller design.

**Remark 3:** Conventional tracking error dynamics (e.g., as seen in [26, 37, 41]), introduce
$\alpha_i$ and $\beta_i$ in the rotation angles $\psi_{ei}$ and $\theta_{ei}$, defined as $\psi_{ei} = \chi_i - \chi_0$ and $\theta_{ei} = v_i - v_0$, where $\chi_i = \psi_i + \beta_i$ and $v_i = \theta_i + \alpha_i$ are rotation angles from frame $\{W_i\}$ to frame $\{I\}$. Note that although this definition is rigorous in a 2D plane, it cannot always be used in 3D space. For instance, by coordinate transformation, one has $R^W_{B_i}(\psi_i, \theta_i) = R^W_{W_i}(\chi_i, \upsilon_i) R^B_{W_i}(-\beta_i, \alpha_i)$. Such a transformation does not always imply that $\chi_i = \psi_i + \beta_i$ nor $v_i = \theta_i + \alpha_i$, due to nonlinearity induced by 3D coordinate transformation. Therefore, this paper instead puts forward kinematic nonlinearity $f_{pi}$, based on equivalent coordinate transformation, by which the effects of $\alpha_i$ and $\beta_i$ are considered in position error dynamics rather than angle error dynamics.

### 2.4. Problem formulation

The formation control objective in this paper is that, when following a leader moving on a smooth trajectory, a group of underactuated AUVs is able to form a desired 3D geometric pattern, regardless of unmeasurable velocities, system constraints, and switching topologies. This general objective can be partitioned into the following two sub-objectives.

The first is a *distributed estimation objective*. Under switching topologies, the information of the leader, e.g., relative position $\mathbf{p}_{ei} = \mathbf{p}_i - \mathbf{p}_0$, might not be known by all followers, therefore the formation tracking error $\mathbf{e}_{pi}$ is not be available for the control design. To address this issue, each follower should estimate the information of the leader in a distributed manner, such that

$$\lim_{t \to \infty} \|\mathbf{n}_0(t) - \mathbf{n}_{00}(t)\| = 0,$$

(13)

where $\mathbf{n}_0$ is the later-defined information of the leader, and $\mathbf{n}_{00}$ represents the estimation of $\mathbf{n}_0$ by the $i$th follower.

The second is a *geometric objective*. Under unmeasured velocities, lumped uncertainties, and system constraints, the AUVs should form a desired formation prescribed by $\mathbf{d}_i$, such that

$$\lim_{t \to \infty} \|\mathbf{e}_{pi}(t)\| < \epsilon_p,$$

(14)

where $\epsilon_p$ is a small positive constant.

In consideration of physical constraints during formation tracking, three practical assumptions are made.

**Assumption 2** [23, 24, 35]: For lumped uncertainties in Eq. (3), there exists a positive constant $\bar{D}$, such that $\|D_{(\cdot)i}\| \leq \bar{D}$ and $\|\dot{D}_{(\cdot)i}\| \leq \bar{D}$.

**Remark 4:** From a practical point of view, lumped uncertainties $D_{(\cdot)i}$ and $\dot{D}_{(\cdot)i}$ are time-varying and unpredictable, but are limited in the sense of energy. Assumption 1 is made for ESOS to estimate uncertainties and uncertainties simultaneously (refer to Theorem 2). However, if velocity is measurable and only uncertainties $D_{(\cdot)i}$ need to be estimated, the boundedness of $\dot{D}_{(\cdot)i}$ can be removed [41]. To determine an accurate upper bound $\bar{D}$, extensive computational fluid dynamics analysis or tower tank experiment analysis should be performed under various vehicle operating conditions [42], therefore it is normally difficult to know $\bar{D}$ beforehand. It is worth noting that the proposed method can be performed without this prior knowledge (refer to Theorem 2).

**Assumption 3** [36]: The graphs are jointly connected across each interval $[t_k, t_{k+1})$. 

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Remark 5: Only joint connectivity is required and the graphs are permitted to be disconnected across some subintervals of time. This is distinct from switching topologies in [32–34], which assume all possible topologies to always be connected.

Assumption 4 [23]: For the leader vehicle, there exists a positive constant $\bar{p}$, such that $|\dot{p}_0| \leq \bar{p}$ and $|\ddot{p}_0| \leq \bar{p}$.

Remark 6: Assumption 4 is reasonable, because in practice the velocities and control inputs are constrained. Moreover, Assumption 3 is made because followers need to estimate the information of the leader under switching topologies, and in this case it is indispensable to assume these states to be bounded. Note that $\dot{p}_0$ and $\ddot{p}_0$ are the desired translational speed and acceleration of the formation. To determine the upper bound $\bar{p}$, the formation maneuver requirements as well as the velocity and control input constraints of each AUV need to be taken into account.

To solve the 3D formation control problem outlined here, a hybrid formation protocol is proposed, as shown in Fig. 3. Firstly, to achieve the distributed estimation objective, a DO is designed for followers (see Sec. 3). Secondly, a compound velocity-free formation tracking control strategy is developed to achieve the geometric objective (see Sec. 4).

3. Distributed observer design

Under the leader-follower structure, the information of the leader must be integrated such that followers can track the motion of the leader and maintain the target geometric pattern. This need has inspired the use of AUV interactions to collaboratively estimate leader information using a distributed observer (DO), under static topologies [43] and switching but connected topologies [44]. Note that, in underwater conditions, it should be assumed that the topology will sometimes be disconnected. Efforts to design a DO under switching jointly connected topologies can be found in [45, 46]. However, the leader in these two approaches [45, 46] was assumed to be a linear system. Note that, in an AUV swarm, the kinematics are nonlinear and the attitudes are not static. This motivates the following contribution: the design of a DO for an AUV swarm under switching jointly connected topologies.

For the DO design, define the information related to the leader as $\eta_{l0} (t) = [p_{cl} (t), s_{l0} (t)]^T$, where $s_{l0} = [\chi_0, v_0, U_0]^T$. Then, assign $\hat{\eta}_{l0} (t) = [\dot{p}_{cl} (t), \dot{s}_{l0} (t)]^T$ to each follower as the estimation of $\eta_{l0} (t)$, where $\dot{s}_{l0} = [\dot{\chi}_{l0}, \dot{v}_{l0}, \dot{U}_{l0}]^T$. Finally, define consensus-based neighborhood information...
estimation error $\mu_{\eta i}$ as
\[
\mu_{\eta i} = \sum_{j=1}^{N} a_{ij} (t) (\hat{\eta}_{i0} - \hat{\eta}_{j0}) + b_i(t) \hat{\eta}_{i0} + \Phi_{\eta i},
\] (15)
where $\Phi_{\eta i} = [\Phi_{p_i}, \Phi_{s_i}]^T$, with $\Phi_{pi} = \sum_{j=1}^{N} a_{ij} (p_j - p_i) + b_i (p_0 - p_i)$ and $\Phi_{si} = -b_i s_0$. Note that $p_0 - p_i$ and $s_0$ are available for a vehicle if and only if $b_i = 1$. Under Assumption 4 and bounded velocities, there exists a positive constant $\gamma_\Phi$ such that $|\dot{\Phi}_{\eta i}| \leq \gamma_\Phi$.

Remark 7: If velocity constraints are known, it can be deduced that $|\dot{p}_i| \leq |R_B^t| |\nu_B| \leq 3(u_{max} + v_{max} + w_{max}) = \hat{p}_{max}$. Assumption 4 further implies that $|\dot{\Phi}_{pi}| \leq (2N + 1) \hat{p}_{max} + \bar{p}$ and $|\dot{\Phi}_{si}| \leq 2 + \bar{p}$. Therefore, a possible solution for $\gamma_\Phi$ is $\gamma_\Phi = (2N + 1) \hat{p}_{max} + 2 (\bar{p} + 1)$.

Then, the DO is proposed as
\[
\hat{\eta}_{i0} = \begin{cases} 
\frac{1}{\alpha_i} \left( -K_\eta \mu_{\eta i} + \sum_{j=1}^{N} a_{ij} \dot{\hat{\eta}}_{j0} - \kappa_1 \text{sgn} (\mu_{\eta i}) - \kappa_2 \text{sgn}^\gamma (\mu_{\eta i}) \right), & \alpha_i > 0 \\
0, & \alpha_i = 0
\end{cases},
\] (16)
where $\alpha_i = \sum_{j=1}^{N} a_{ij} + b_i$, $K_\eta \in \mathbb{R}^{6 \times 6}$ is a diagonal positive definite matrix, $\kappa_1$ is a positive constant to be determined, and $\kappa_2 > 0$, $\varsigma$ satisfies $0 < \varsigma = \varsigma_1 / \varsigma_2 < 1$ with $\varsigma_1$ and $\varsigma_2$ being positive odd integers.

The observer is implemented in a cooperative manner as follows. If the $j$th follower is the neighbor of the $i$th AUV, $\hat{\eta}_{j0}$ and $\hat{\eta}_{j0}$ are sent to the $i$th follower via a given underwater communication technique. Meanwhile, the sensor of the $i$th AUV can measure the relative position $p_i - p_j$. If the leader is the neighbor of the $i$th follower, $s_0$ and $p_i - p_0$ will be available. Subsequently, the $i$th follower can calculate $\mu_{\eta i}$ with local information, and update the observer as in Eq. (16). In this sense, the proposed DO is a coordination mechanism in which AUVs directly cooperate using local inter-AUV communication and local measurements.

Theorem 1: Let $\kappa_1 \geq \gamma_\Phi$. If Assumption 3 and Assumption 4 hold, with the proposed DO in Eq. (16), then the followers can estimate the information of the leader accurately under switching topologies, i.e., the estimation error $\dot{\eta}_{i0}(t) = \dot{\eta}_{i0}(t) - \eta_{i0}(t)$ converges to zero in finite time. (The proof of Theorem 1 is given in Appendix A.)

With the proposed DO Eq. (16), each follower is able to estimate the information of the leader in finite time denoted by $T_F$. That is, $\eta_{i0}(t) = \hat{\eta}_{i0}(t)$ for $t \geq T_F$. Thus, we can employ $\hat{\eta}_{i0}(t)$ directly in the formation tracking control strategy.

4. Formation tracking control strategy

In this section, a compound velocity-free formation tracking control strategy is designed to achieve the geometric objective. First, a cascaded extended state observer (CESO) is proposed to recover unmeasurable velocities and approximate dynamic uncertainties. Second, an improved 3D nonlinear guidance law is developed at the kinematic level. Third, a constrained robust controller is designed at the dynamic level to guarantee the performance of disturbance rejection as well as stability under constraints.

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4.1. Cascaded ESO design

The ESO is an essential method in active disturbance rejection control and requires minimal information about a dynamic system, thus it has already been used for AUVs in a horizontal plane [23, 35]. However, this conventional ESO cannot directly be used for AUVs in 3D space, for the following two reasons. (i) The kinematics in the yaw direction do not satisfy the integral-connected form. (ii) The conventional ESO requires absolute position measurements, which cannot be guaranteed to be feasible for AUVs in deep water. Therefore, this paper designs a cascaded ESO (CESO) for AUVs in 3D space.

As the first step, for pitch direction, the following ESO is designed to estimate \( q_i \) and \( D_{qi} \):

\[
\begin{align*}
\dot{q}_i &= -k_{q1} (\theta_i - \dot{\theta}_i) + \dot{q}_i \\
\dot{\theta}_i &= -k_{q2} (\theta_i - \dot{\theta}_i) + \tau_i/m_{qi} + \dot{D}_{qi}, \\
\dot{D}_{qi} &= -k_{q3} (\theta_i - \dot{\theta}_i)
\end{align*}
\]

(17)

where \( \dot{\theta}_i, \dot{q}_i, \) and \( \dot{D}_{qi} \) are the respective estimations of \( \theta, q_i, \) and \( D_{qi} \). The constants \( k_{q1}, k_{q2}, \) and \( k_{q3} \) are positive, and to be determined.

In the second step, in order to construct the integral chain form in yaw direction, a state transformation is performed. Let \( \xi_{ri} = \psi_i \cos \theta_i \), then the following ESO to estimate \( r_i \) and \( D_{ri} \) can be designed:

\[
\begin{align*}
\dot{\xi}_{ri} &= -k_{r1} (\xi_{ri} - \xi_{ri}) + \dot{r}_i - \dot{\xi}_{ri} \psi_i \sin \theta_i \\
\dot{r}_i &= -k_{r2} (\xi_{ri} - \xi_{ri}) + \tau_i/m_{ri} + \dot{D}_{ri}, \\
\dot{D}_{ri} &= -k_{r3} (\xi_{ri} - \xi_{ri})
\end{align*}
\]

(18)

where \( \dot{\xi}_{ri}, \dot{r}_i, \) and \( \dot{D}_{ri} \) are the respective estimations of \( \xi_{ri}, r_i, \) and \( D_{ri} \). The constants \( k_{r1}, k_{r2}, \) and \( k_{r3} \) are positive, and to be determined.

In the third step, instead of employing absolute position, the relative position to the leader \( p_{ci} \) is utilized and an equivalent coordinate transformation is performed. \( p_{ci} \) can be expressed in frame \( \{B_i\} \) as \( \xi_{pi} = R_{I}^{B_i} \cdot p_{ci} \) and the dynamics can then be derived as follows:

\[
\begin{align*}
\dot{\xi}_{pi} &= S_{I}^{B_i} (r_i, q_i, \theta_i) \xi_{pi} + \nu_{Bi} - R_{I}^{B_i} \cdot \dot{p}_0 \\
\dot{\nu}_{Bi} &= \tau_{pi} + D_{pi}
\end{align*}
\]

(19)

where \( \tau_{pi} = [\tau_{ui}/m_{ui}, 0, 0]^{T} \) and \( D_{pi} = [D_{ui}, D_{vi}, D_{wi}]^{T} \). \( R_{I}^{B_i} = \left( R_{I}^{B_i} (\psi_i, \theta_i) \right)^{T} \) is the rotation matrix from frame \( \{I\} \) to frame \( \{B_i\} \) and \( S_{I}^{B_i} (r_i, q_i, \theta_i) \) is defined as

\[
S_{I}^{B_i} (r_i, q_i, \theta_i) = \begin{bmatrix} 0 & r_i & -q_i \\ -r_i & 0 & -r_i \tan \theta_i \\ q_i & r_i \tan \theta_i & 0 \end{bmatrix}.
\]

(20)

Based on the dynamics given in Eq. (19), the following ESO is designed to estimate \( \nu_{Bi} \) and \( D_{pi} \):

\[
\begin{align*}
\dot{\xi}_{pi} &= -K_{p1} (\xi_{pi} - \xi_{pi}) + S_{I}^{B_i} (\dot{r}_i, \dot{q}_i, \dot{\theta}_i) \xi_{pi} + \nu_{Bi} - R_{I}^{B_i} \cdot \dot{p}_0 \\
\dot{\nu}_{Bi} &= -K_{p2} (\xi_{pi} - \xi_{pi}) + D_{pi} + \tau_{pi} \\
\dot{D}_{pi} &= -K_{p3} (\xi_{pi} - \xi_{pi})
\end{align*}
\]

(21)
where \( K_{p1} \in \mathbb{R}^{3 \times 3}, K_{p2} \in \mathbb{R}^{3 \times 3} \) and \( K_{p3} \in \mathbb{R}^{3 \times 3} \) are diagonal positive definite matrices; and \( \hat{\xi}_{pi}, \hat{\nu}_{Bi} = [\hat{u}_i, \hat{v}_i, \hat{w}_i]^T \), and \( \hat{D}_{pi} = [\hat{D}_{ui}, \hat{D}_{vi}, \hat{D}_{wi}]^T \) are respective estimations of \( \xi_{pi}, \nu_{Bi} \), and \( D_{pi} \). \( \xi_{pi} \) and \( \hat{p}_0^i \) are defined based on the estimation results of the DO and take the form

\[
\begin{align*}
\xi_{pi} &= R_i^B \hat{p}_i
\end{align*}
\]

\[
\begin{align*}
\hat{p}_0^i &= \hat{U}_i [ \cos \hat{\chi}_{i0} \cos \hat{\nu}_{i0}, \sin \hat{\chi}_{i0} \cos \hat{\nu}_{i0}, -\sin \hat{\nu}_{i0} ]^T.
\end{align*}
\]

Let the velocity estimation error vector be \( \tilde{\nu}_i = [\tilde{u}_i, \tilde{v}_i, \tilde{w}_i]^T \) and the uncertainty estimation error vector be \( \tilde{D}_i = [\tilde{D}_{ui}, \tilde{D}_{vi}, \tilde{D}_{wi}, \tilde{D}_{qi}, \tilde{D}_{ri}]^T \), where \( \tilde{\rho}_i = \tilde{\rho}_i - \rho_i \) and \( \tilde{D}_{pi} = \tilde{D}_{pi} - D_{pi} \), \( (\rho = u, v, w, q, r) \). Then the CESO Eqs. (17, 18, 21) has the following property.

**Theorem 2**: Considering the CESO Eqs. (17, 18, 21) under Assumptions 1–4, \( \tilde{\nu}_i \) and \( \tilde{D}_i \) are ultimately bounded. (The proof of Theorem 2 is given in Appendix B.)

### 4.2. Improved 3D kinematic guidance law

Line-of-sight (LOS) guidance is commonly used in motion control of marine vehicles, as it can convert the tracking errors of the underactuated DOF into controllable orientation errors. However, conventional LOS has limited robustness against external disturbances. To enhance the performance, improved LOS guidance laws were developed, as summarized in [47]. However, most methods in [47] only considered a 2D plane. In the research, the existing methods are extended to 3D space.

Before proceeding with the guidance law, define the estimation-based information needed for it as

\[
\begin{align*}
\tilde{e}_{pi} &= [\tilde{e}_{xi}, \tilde{e}_{yi}, \tilde{e}_{zi}]^T = (R_F^i (\hat{\chi}_{i0}, \hat{\nu}_{i0}))^T (\hat{p}_i - d_i) \\
\tilde{U}_i &= \sqrt{\tilde{u}_i^2 + \tilde{v}_i^2 + \tilde{w}_i^2} \\
\tilde{\nu}_{W_i} &= [\tilde{U}_i, 0, 0]^T \\
\tilde{f}_{pi} &= [\tilde{f}_{xi}, \tilde{f}_{yi}, \tilde{f}_{zi}]^T = (R_F^i (\hat{\chi}_{i0}, \hat{\nu}_{i0}))^T R_B^i (\hat{\nu}_{B_i} - \hat{\nu}_{W_i})
\end{align*}
\]

where \( \tilde{e}_{pi}, \tilde{U}_i, \tilde{\nu}_{W_i}, \) and \( \tilde{f}_{pi} \) are the respective estimations of \( e_{pi}, U_i, \mu_{W_i}, \) and \( f_{pi} \). Based on the tracking error dynamics in Eq. (12), and as needed for LOS guidance, define the desired rotation angles \( \psi_{ei}^d \) and \( \theta_{ei}^d \) of rotation matrix \( R_B^i (\psi_{ei}^d, \theta_{ei}^d) \), such that

\[
\begin{align*}
\psi_{ei}^d &= \tan^{-1} (-(\tilde{e}_{yi} + \delta_{yi})/\Delta_y) \\
\theta_{ei}^d &= \tan^{-1} ((\tilde{e}_{zi} + \delta_{zi})/\Delta_z)
\end{align*}
\]

where \( \Delta_y \) and \( \Delta_z \) can be termed look-ahead distances, satisfying \( \Delta_z = \sqrt{\Delta_y^2 + (\tilde{e}_{yi} + \delta_{yi})^2} \). The auxiliary terms \( \delta_{yi} \) and \( \delta_{zi} \) are designed to compensate respectively for the nonlinearities \( f_{yi} \) and \( f_{zi} \), and take the form

\[
\begin{align*}
\delta_{yi} &= \frac{\tilde{e}_{yi}(\gamma_{yi})^2 + \gamma_{yi}\sqrt{\Delta_y^2(1 - \gamma_{yi})} + \tilde{e}_{yi}^2}{1 - \gamma_{yi}^2}, \\
\delta_{zi} &= \frac{\tilde{e}_{zi}(\gamma_{zi})^2 + \gamma_{zi}\sqrt{\Delta_z^2(1 - \gamma_{zi})} + \tilde{e}_{zi}^2}{1 - \gamma_{zi}^2}
\end{align*}
\]
where \( \gamma_{yi} = \hat{f}_{yi} \ddot{U}_i \) and \( \gamma_{zi} = \hat{f}_{zi} \ddot{U}_i \). To steer the rotation angles \( \psi_{ei} \) and \( \theta_{ei} \) to the desired angles \( \psi_{ei}^d \) and \( \theta_{ei}^d \), recall

\[
R^\text{F}_{\dot{X}_0} (\dot{X}_0, \dot{\gamma}_0) R^\text{F}_{\dot{X}_0} \left( \psi_{ei}^d, \theta_{ei}^d \right) = R^\text{F}_{\dot{X}_0} \left( \psi_{ei}, \theta_{ei} \right),
\]

where \( \theta_{ei}^d \) and \( \psi_{ei}^d \) are the desired pitch and yaw angles of the vehicles, respectively. Expanding Eq. (26) yields

\[
\begin{align*}
\theta_{ei}^d &= \arcsin \left( \sin \dot{\gamma}_{0i} \cos \theta_{ei}^d \cos \psi_{ei}^d + \cos \dot{\gamma}_{0i} \sin \theta_{ei}^d \right) \\
\psi_{ei}^d &= \arctan \left( \frac{\psi_{ei}^d}{\theta_{ei}^d} \right),
\end{align*}
\]

such that

\[
\begin{align*}
\psi_{ei}^d &= \sin \dot{\gamma}_{0i} \cos \dot{\gamma}_{0i} \cos \psi_{ei}^d \cos \theta_{ei}^d + \cos \dot{\gamma}_{0i} \sin \psi_{ei}^d \cos \theta_{ei}^d \\
&\quad - \sin \dot{\gamma}_{0i} \sin \dot{\gamma}_{0i} \sin \theta_{ei}^d \\
\psi_{ei}^d &= \cos \dot{\gamma}_{0i} \cos \dot{\gamma}_{0i} \cos \psi_{ei}^d \cos \theta_{ei}^d - \sin \dot{\gamma}_{0i} \sin \psi_{ei}^d \cos \theta_{ei}^d \\
&\quad - \cos \dot{\gamma}_{0i} \sin \dot{\gamma}_{0i} \sin \theta_{ei}^d
\end{align*}
\]

Note that if \( \psi_{ei} = \psi_{ei}^d \) and \( \theta_{ei} = \theta_{ei}^d \) under the kinematic controller, then \( \psi_{ei} = \psi_{ei}^d \) and \( \theta_{ei} = \theta_{ei}^d \) will be guaranteed. To achieve this geometric objective, the kinematic controller is designed as

\[
\begin{align*}
u_i^d &= \left( -b_x \dot{e}_{x_i} + \dot{U}_i - \hat{f}_{x_i} \right) \cos \alpha_i \cos \beta_i \Delta_i \\
\theta_i^d &= -b_y \left( \theta_{ei} - \theta_{ei}^d \right) + \dot{\theta}_{ei}^d \\
\psi_i^d &= \left( -b_{\psi} \left( \psi_{ei} - \psi_{ei}^d \right) + \dot{\psi}_{ei}^d \right) \cos \theta_{ei}
\end{align*}
\]

where \( b_x, b_y \) and \( b_{\psi} \) are positive constants and \( \Delta_i = \sqrt{\Delta_y^2 + (\dot{\xi}_{yi} + \delta_{yi})^2 + (\dot{\xi}_{zi} + \delta_{zi})^2} \). \( \Delta_y \).

**Remark 8**: From Eq. (29), the derivatives of command signals are required for the proposed kinematic controller to be implemented. They could be derived via analytic calculation, but doing so would carry a high computational cost due to the complex form. Therefore, the command filter according to [16] is employed to approximate the derivatives, as follows:

\[
\dot{\theta}_{ei}^d = \frac{w_n^2}{s^2 + 2\xi w_n s + w_n^2} \theta_{ei}^d,
\]

where \( \dot{\theta}_{ei}^d \) is the estimation of \( \dot{\theta}_{ei}^d \), and \( w_n > 0 \) and \( \xi > 0 \) are the natural frequency and damp ratios. If \( \xi \) is set appropriately and \( w_n \) is sufficiently large, then a small approximation error is ensured. The approximation error is not addressed here, because it is beyond the scope of this paper. For reference, a detailed discussion on this topic can be found in [48]. The bounded approximation error can be included in the uncertainty \( M_{\alpha i} \) in Eq. (D.2) and has no effect on the main result.

### 4.3. Constrained robust dynamic controller

Using the proposed 3D kinematic guidance law, the desired velocities can be derived based on the estimation results. However, these command signals cannot ensure that the velocity constraints and control input constraints are well satisfied. Therefore, a constrained robust controller is designed at the dynamics level, wherein two anti-windup compensators are introduced to address
saturations and potential unstable behavior. Due to the similarity of the dynamics models of $\tau_{ui}$, $\tau_{qi}$, and $\tau_{ri}$, only the design process details for $\tau_{qi}$ in the pitch direction will be given in detail.

Firstly, note that $q_i^d$ may violate the velocity constraint. To keep the command signal within the constraint, design the velocity command truncation as

$$\Delta q_i = \text{sat} \left( q_i^d - \phi_{qi} \right) - \left( q_i^d - \phi_{qi} \right),$$

where $\phi_{qi}$ is as an anti-windup compensator to regulate the velocity command signal, to be designed later.

Secondly, the control input is intrinsically subject to saturation phenomena. To keep the control input from reaching saturation, design the control input truncation as

$$\Delta \tau_{qi} = \tau_{qi} - \tau_{qi}^s = \text{sat} \left( \tau_{qi} \right) - \tau_{qi}^s,$$

where $\phi_{\tau qi}$ is an anti-windup compensator to regulate the control input, to be designed later. Then, the following constrained dynamic controller is proposed:

$$\tau_{qi} = m_{qi} \left( -c_q (s_{qi} - \phi_{\tau qi}) + \dot{q}_i^d - \bar{D}_{qi} - \dot{\phi}_{qi} \right),$$

where $s_{qi} = q_i - q_i^d + \phi_{qi}$ is the regulated velocity tracking error and $c_q$ is a positive constant, to be determined. $\dot{q}_i^d$ can be calculated by command filter Eq. (30). The adaptive laws for $\phi_{qi}$ and $\phi_{\tau qi}$ are as follows:

$$\dot{\phi}_{qi} = \begin{cases} - \left( w_1 + \frac{w_2 \Delta q_i^2}{2|\phi_{qi}|^2} \right) \phi_{qi} + w_2 \Delta q_i, |\phi_{qi}| \geq \mu_q \\ 0, |\phi_{qi}| < \mu_q \end{cases}$$

and

$$\dot{\phi}_{\tau qi} = \begin{cases} - \left( l_1 + \frac{|g_{\tau qi}| + \frac{l_2 \Delta \tau_{qi}^2}{|\phi_{\tau qi}|^2}}{2|\phi_{\tau qi}|^2} \right) \phi_{\tau qi} + l_2 \Delta \tau_{qi}, |\phi_{\tau qi}| \geq \mu_{\tau q} \\ 0, |\phi_{\tau qi}| < \mu_{\tau q} \end{cases}$$

where $w_1$, $w_2$, $l_1$, and $l_2$ are positive constants, to be determined; $\mu_q$ and $\mu_{\tau q}$ are designed small positive constants; and $g_{\tau qi} = s_{qi} \Delta \tau_{qi}/m_{qi}$.

**Theorem 3**: Let $c_q > 0$, $2w_1 > w_2$, and $2l_1 > c_q + l_2$, under Assumptions 1–4, with the proposed constrained robust dynamic controller Eq. (33) and the anti-windup compensators Eq. (34) and Eq. (35), then it follows that $s_{qi}$, $\phi_{qi}$, and $\phi_{\tau qi}$ are ultimately bounded. (The proof of Theorem 3 is given in Appendix C.)

Let velocity command tracking error $e_{qi} = q_i - q_i^d$. It follows from Theorem 2 and Theorem 3 that

$$|e_{qi}| = |s_{qi} - \tilde{q}_i - \phi_{qi}| \leq |s_{qi}| + |\phi_{qi}| + |\dot{q}_i|,$$

thus, $e_{qi}$ is guaranteed to be ultimately bounded.
5. Stability analysis of the closed-loop system

This section establishes the stability of the closed-loop system consisting of the proposed formation protocol and \( N \) formation tracking error systems, which also establishes the achievement of the geometric objective, proceeding from Theorem 4, as follows.

**Theorem 4:** Considering the switching topology \( \mathcal{G}(t) \) and the formation tracking error system Eq. (12), under Assumptions 1–4, with the proposed formation protocol consisting of DO Eq. (16), CESO Eqs. (17, 18, 21), kinematic controller Eq. (29), and dynamic controller Eq. (33) with compensators Eq. (34) and Eq. (35), then it follows that the formation tracking error \( \mathbf{e}_{pi} \) is ultimately bounded. (The proof of Theorem 4 is given in Appendix D.)

6. Simulation results

The performance of the proposed formation protocol is evaluated in numerical simulation and compared to three state-of-the-art control methods. Moreover, two simulation examples are performed to substantiate the effectiveness of the proposed method on large-scale AUV swarm.

6.1. Simulation setup

The simulated experiments are conducted with a group of underactuated AUVs comprising one leader and six followers, with model parameters replicated from [41]. The constraints are enforced as \( u_{\text{max}} = -u_{\text{min}} = 1.2, \quad q_{\text{max}} = r_{\text{max}} = -q_{\text{min}} = -r_{\text{min}} = 0.4, \) and \( \tau_{\rho}^{\text{max}} = -\tau_{\rho}^{\text{min}} = 200 \) (\( \rho = u, q, r \)). By default, the variable values are in SI units; hereafter units are omitted from the notation.

The trajectory of the leader is predefined as

\[
p_0(t) = \left[ 0.4t, 10 + 10 \sin\left(\frac{\pi}{100} t\right), -20 + 10 \sin\left(\frac{\pi}{100} t\right) \right]^T, \tag{37}
\]

and the desired formation is specified as

\[
\begin{align*}
\mathbf{h}_1 &= [-4 \cos(\pi/12) \cos(\pi/6), 4 \cos(\pi/12) \sin(\pi/6), 4 \sin(\pi/12)]^T \\
\mathbf{h}_2 &= [-4 \cos(\pi/12) \cos(2\pi/3), -4 \cos(\pi/12) \sin(2\pi/3), -4 \sin(\pi/12)]^T \\
\mathbf{h}_3 &= [-8 \cos(\pi/12) \cos(\pi/6), 8 \cos(\pi/12) \sin(\pi/6), 8 \sin(\pi/12)]^T \\
\mathbf{h}_4 &= [-8 \cos(\pi/12) \cos(2\pi/3), -8 \cos(\pi/12) \sin(2\pi/3), -8 \sin(\pi/12)]^T \\
\mathbf{h}_5 &= [-12 \cos(\pi/12) \cos(\pi/6), 12 \cos(\pi/12) \sin(\pi/6), 12 \sin(\pi/12)]^T \\
\mathbf{h}_6 &= [-12 \cos(\pi/12) \cos(2\pi/3), -12 \cos(\pi/12) \sin(2\pi/3), -12 \sin(\pi/12)]^T
\end{align*}
\]

The initial positions of the followers are set away from the leader vehicle, given by \( \mathbf{p}_1(0) = [-10, 15, -5]^T, \mathbf{p}_2(0) = [5, 3, -5]^T, \mathbf{p}_3(0) = [5, 5, -5]^T, \mathbf{p}_4(0) = [-5, 0, -5]^T, \mathbf{p}_5(0) = [-10, -5, -5]^T, \mathbf{p}_6(0) = [-10, 2, -5]^T \). The initial orientations are set to zero. The dynamic uncertainties, consisting of second-order nonlinear hydrodynamic damping terms and environmental disturbances, are simulated as

\[
\begin{align*}
\tau_{dui} &= -d_{ui|ui} |u_i| u_i + 0.2m_{ui} \sin(0.3t) \\
\tau_{dvi} &= -d_{vi|vi} |v_i| v_i + 0.05m_{vi} \sin(0.2t) \\
\tau_{dwi} &= -d_{wi|wi} |w_i| w_i + 0.05m_{wi} \sin(0.2t) \\
\tau_{dzi} &= -d_{zi|zi} |q_i| q_i + 0.2m_{zi} \sin(0.3t) \\
\tau_{dri} &= -d_{ri|ri} |r_i| r_i + 0.2m_{ri} \sin(0.3t)
\end{align*}
\]
where $d(\cdot)_{i|(\cdot)_{i}}$ are damping parameters. In order to test the robustness of the proposed method, certain variations are added to these damping terms, given by $d(\cdot)_{i|(\cdot)_{i}} = \{200, 133, 100, 80, 66, 57\}$. The simulations are conducted in Matlab 2016b with the solver ode3(Bogacki-Shampine) and a fixed step size of 0.01s, on a PC with an Intel i5-7200 CPU, with 8GB RAM, and running Win 10 64-bit OS.

6.2. Four methods for comparison

| Method 1 | Relative | Switching Jointly | DO | CESO | Improved LOS | AC | - |
| Method 2 | Relative | Switching | DO | CESO | Conventional LOS | RG | [17] |
| Method 3 | Absolute | Static | Known | Measured | Sliding-based | - | [28] |
| Method 4 | Relative | Static | Known | Measured | PP | - | [27] |

Notations: DO = Distributed Observer; CESO = Cascaded Extended State Observer; LOS = Line-of-Sight; AC = Anti-windup Compensator; RG = Reference Governor; PP = Prescribed Performance-based

Table 3: Parameters of the proposed Method 1 (also partly utilized in Method 2).

<table>
<thead>
<tr>
<th>Components</th>
<th>Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>DO Eq. (16)</td>
<td>$K_{\eta} = 154\eta$, $\kappa_{1} = 8$, $\kappa_{2} = 0.1$, $\zeta = 7/9$</td>
</tr>
<tr>
<td>CESO Eqs. (17, 18, 21)</td>
<td>$k_{q1} = k_{r1} = 60$, $k_{q2} = k_{r2} = 1200$, $k_{q3} = k_{r3} = 8000$, $K_{p1} = 30I_{3}$, $K_{p2} = 300I_{3}$, $K_{p3} = 1000I_{3}$</td>
</tr>
<tr>
<td>LOS guidance law Eq. (24)</td>
<td>$\Delta_{q} = 4$</td>
</tr>
<tr>
<td>Kinematic law Eq. (29)</td>
<td>$b_{x} = b_{y} = b_{\psi} = 0.3$, $\xi = 2$, $w_{m} = 10$</td>
</tr>
<tr>
<td>Dynamic laws</td>
<td>$w_{1} = 30$, $w_{2} = 50$, $l_{1} = 10$, $l_{2} = 1e - 5$</td>
</tr>
<tr>
<td>Eqs. (33), (34), and (35)</td>
<td>$c_{p} = 2$, $\mu_{p} = 0.01$, $\mu_{r_{p}} = 1$, $(\rho = u, q, r)$</td>
</tr>
</tbody>
</table>

Figure 4: Switching jointly connected communication topologies. (a) Three disconnected topologies. (b) Switching signal $\sigma (t)$ in simulation.

The proposed formation protocol (Method 1) is compared with three alternative protocols (Methods 2-4), as summarized in Table 2.

**Method 1:** The proposed formation protocol. The controller parameters used in the experiments are presented in Table 3. To test the performance under switching jointly connected networks, the communication topologies in the experiments switch between $\bar{G}_{1}$, $\bar{G}_{2}$, and $\bar{G}_{3}$, according to signal $\sigma (t)$, as shown in Fig. 4.

**Method 2:** A distributed velocity-free tracking controller with conventional LOS guidance, incorporating a reference governor (RG) to address system constraints. The controller parameters
used in the experiments are the same values as those used for Method 1, except for those of the RG, which are set according to [17]. The topologies in the experiments switch between $\bar{G}_1 \cup \bar{G}_2$ and $\bar{G}_2 \cup \bar{G}_3$, and each phase is active for 10s.

**Method 3:** A centralized formation control approach [27], wherein each follower connects directly (and only) to the leader vehicle. This formation protocol was designed based on integral terminal sliding mode control. The parameters are tuned according to [27].

**Method 4:** A distributed formation approach proposed in [28], wherein the controller was designed by concerning the prescribed transient and steady-state control performance. The controller parameters are tuned according to [28], and the topology is $\bar{G}_1 \cup \bar{G}_2$.

### 6.3. Results

The simulation results are presented in Figs. 5–11, where $F_i \ (i = 1, 2, \ldots, 6)$ is the $i$th follower. The control performance of the four methods is assessed according to maneuverability performance, tracking accuracy, and constraint satisfaction. Additionally, the estimation accuracies of the proposed DO and CESO in Method 1 are assessed, with a comparison to an existing DO from the literature.

**Maneuverability performance:** Fig. 5 shows the formation evolution of the AUV swarm under each method. Methods 1, 3, and 4 each achieve smooth formation maneuvers, while Method 2 leads
to shaking. The observed difference between Methods 1 and 2 occurs because the conventional LOS guidance law in Method 2 has limited robustness against nonzero sideslip and attack angle, while the improved LOS guidance law in Method 1 achieves smoother maneuvers by introducing compensation terms.

Tracking accuracy: Fig. 6 shows the evolution of formation tracking errors $e_{pi}$ under each method. The convergence rates of all methods are similar, with Methods 3 and 4 converging slightly more quickly (with Method 1 being slightly worse than Method 2). The zoom-in inserts show that Method 2 leads to noticeably larger steady state errors than the other methods and that Method 1 has the smallest steady state errors (slightly better than Methods 3 and 4).
Table 4: Performance comparison of the four methods.

<table>
<thead>
<tr>
<th>i-th AUV</th>
<th>IAE = $\int |e_{pi}|dt$ ($\times 10^2$)</th>
<th>ITAE = $\int t |e_{pi}|dt$ ($\times 10^4$)</th>
<th>RMSE = $\left(\int |e_{pi}|^2 / T\right)^{0.5}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>M1</td>
<td>1.008</td>
<td>4.207</td>
<td>0.263</td>
</tr>
<tr>
<td>M2</td>
<td>2.991</td>
<td>1.098</td>
<td>0.027</td>
</tr>
<tr>
<td>M3</td>
<td>1.462</td>
<td>1.029</td>
<td>0.125</td>
</tr>
<tr>
<td>M4</td>
<td>1.804</td>
<td>1.544</td>
<td>0.204</td>
</tr>
<tr>
<td>M1</td>
<td>1.300</td>
<td>4.435</td>
<td>0.329</td>
</tr>
<tr>
<td>M2</td>
<td>4.366</td>
<td>4.680</td>
<td>0.571</td>
</tr>
<tr>
<td>M3</td>
<td>2.146</td>
<td>1.098</td>
<td>0.027</td>
</tr>
<tr>
<td>M4</td>
<td>2.620</td>
<td>1.572</td>
<td>0.119</td>
</tr>
<tr>
<td>M1</td>
<td>1.618</td>
<td>4.715</td>
<td>0.337</td>
</tr>
<tr>
<td>M2</td>
<td>3.600</td>
<td>1.067</td>
<td>0.028</td>
</tr>
<tr>
<td>M3</td>
<td>1.872</td>
<td>1.38</td>
<td>0.116</td>
</tr>
<tr>
<td>M4</td>
<td>2.023</td>
<td>1.16</td>
<td>0.16</td>
</tr>
<tr>
<td>M1</td>
<td>4.519</td>
<td>5.163</td>
<td>0.780</td>
</tr>
<tr>
<td>M2</td>
<td>6.188</td>
<td>1.239</td>
<td>0.029</td>
</tr>
<tr>
<td>M3</td>
<td>4.090</td>
<td>1.641</td>
<td>0.114</td>
</tr>
<tr>
<td>M4</td>
<td>4.090</td>
<td>0.204</td>
<td>0.156</td>
</tr>
<tr>
<td>M1</td>
<td>3.824</td>
<td>5.143</td>
<td>0.691</td>
</tr>
<tr>
<td>M2</td>
<td>5.726</td>
<td>1.156</td>
<td>0.029</td>
</tr>
<tr>
<td>M3</td>
<td>3.192</td>
<td>1.693</td>
<td>0.115</td>
</tr>
<tr>
<td>M4</td>
<td>3.192</td>
<td>0.028</td>
<td>0.188</td>
</tr>
<tr>
<td>AVG</td>
<td>2.567</td>
<td>4.266</td>
<td>0.495</td>
</tr>
<tr>
<td></td>
<td>2.172</td>
<td>1.111</td>
<td>0.028</td>
</tr>
<tr>
<td></td>
<td>2.559</td>
<td>1.592</td>
<td>0.177</td>
</tr>
</tbody>
</table>

In Table 4, tracking accuracy is also compared in terms of Integral Absolute Error (IAE), Integral Time Absolute Error (ITAE) for the entire evolution, and Root Mean Square Error (RMSE) for the steady phase. Method 3 has the lowest average IAE value, benefiting from its faster transient response. However, in the case of ITAE and RMSE, Method 1 preserves the lowest error values due to its minimal steady state errors. Method 2 has the highest error values for all three indicators.
Constraint satisfaction: The profiles of velocity and control inputs in the surge direction (i.e., $u_i$ and $\tau_{ui}$ respectively) are given in Figs. 7 and 8. Methods 1 and 2 are capable of achieving velocity satisfaction, while both Methods 3 and 4 violate the velocity constraints at the initial stage, as seen in the zoom-in inserts in Fig. 7. This corresponds to Methods 3 and 4 having faster convergence rates than Methods 1 and 2. The input profiles in Fig. 8 show that the surge inputs in all four methods inevitably saturate due to the large initial tracking error.

Estimation accuracy: The estimation performance of the proposed DO in Method 1 is presented in Fig. 9. Using the DO of Method 1, the leader’s information can be estimated by Follower 1 within a few seconds, so that the distributed estimation objective is achieved. For comparison, an existing DO [43] is also evaluated in Fig. 10, under the same topologies used in the experimental conditions of Method 1 (i.e., under switching jointly connected topologies). In Fig. 10, only the estimation signals of the first five seconds are provided, because these signals will diverge with $t$ increasing.

The DO of [43] deteriorates under a lack of topology connectivity, as might be present in underwater communications, while the proposed DO of Method 1 achieves the estimation objective under the same conditions.
Fig. 11 shows the estimation results of the proposed CESO in Method 1. It can be seen that
the unmeasurable velocities and unknown disturbances can be precisely recovered, and therefore
velocity sensors are not required.

![Figure 9: Estimation result of the leader’s information (a) $p_{e_1}$ and (b) $s_0$ by Follower 1 using the proposed DO in Method 1.](image)

Figure 9: Estimation result of the leader’s information (a) $p_{e_1}$ and (b) $s_0$ by Follower 1 using the proposed DO in Method 1.

![Figure 10: Estimation result of leader’s information (a) $p_0$ and (b) $s_0$ via an existing DO in the literature [43] under switching jointly connected topologies (i.e., the same topology conditions under which Method 1 is tested).](image)

Figure 10: Estimation result of leader’s information (a) $p_0$ and (b) $s_0$ via an existing DO in the literature [43] under switching jointly connected topologies (i.e., the same topology conditions under which Method 1 is tested).
6.4. Scalability of the proposed method

Scalability is an important advantage of distributed control strategies. In order to substantiate the scalability of our proposed protocol, we perform two simulations with larger robot swarms: a swarm of 27 AUVs and a swarm of 100 AUVs.

In the first scenario, the target formation is a cube with 27 nodes and 52 edges. The initial position and trajectory of the leader AUV are those defined in Sec. 6.1. The initial positions of the followers are distributed in a grid on the surface of the sea. As can be seen in Fig. 12(a), the target formation can be achieved within 100 s, and the formation tracking errors converge to the neighborhood around the origin, as presented in Fig. 12(b).

In the second scenario, the target formation is a cuboid with $5 \times 5 \times 4$ nodes and 235 edges. The initial positions of the followers are distributed in a 30 m radius circle around the leader. The leader moves straight forward, at a linear velocity of 0.5 m/s. The resulting trajectories of all followers in Fig. 13(a) show that our proposed formation control method can steer a swarm of one hundred AUVs to achieve the target formation. As can be seen in Fig. 13(b), which shows the formation tracking error of 25 arbitrary followers, the target formation can be achieved within 150 s and the formation tracking errors converge to the neighborhood around the origin.

We can notice that, in each scenario, the convergence rate is similar for all followers. This occurs because our proposed DO decomposes the formation control problem of the AUV swarm into decentralized trajectory tracking using local inter-AUV communication and local measurements. Thus, the convergence speed is mainly determined by the guidance-based kinematic control law, as discussed in Appendix D.

6.5. Discussion

The following conclusions can be drawn from the simulation results.

1. Compared with Method 2, the proposed Method 1 guarantees smooth formation evolution and minimal steady state errors, which substantiates the improved performance of the proposed LOS guidance law in Method 1.
2. Compared with Methods 3 and 4, the proposed Method 1 performs better in terms of steady state errors and constraint satisfaction. In addition, the proposed CESO in Method 1 removes dependence on velocity sensors and the assumption of static topologies, which cannot be guaranteed in practice.

3. Compared with the existing DO in [43], Method 1 is verified to be applicable under switching jointly connected topologies. The proposed DO in Method 1 removes dependence on the common assumption of constant connectivity, which cannot be guaranteed in practice.

It can be concluded that the proposed formation protocol is proved to have the desired maneuverability performance, improved tracking accuracy, and guaranteed constraint satisfaction, in comparison to the state of the art. Moreover, the proposed DO and CESO handle several practical constraints for formation control of underactuated AUVs that have not previously been fully considered: switching jointly connected topologies and unavailable velocity measurements. Regarding scalability and large-scale swarms, it should be noted that the simulation experiments in this research were performed in an ideal environment. For instance, time delays in communication techniques such as acoustic communication in real oceanic environments are not considered in this research, but are inevitable in practice. As the group size grows, the adverse influence of the delay factor will increase. In addition, collision avoidance is an important issue for large-scale
robot swarms, posing substantial challenges in formation achievement and maintenance. Addressing these challenges for deployment of large-scale AUV swarms will require future research or extensions, for instance using multi-leader protocols.

**Remark 9:** This research firstly defines some of the research gaps in formation control of AUV swarms by classifying existing approaches in terms of topology, measurements, constraints, and dimensions, as shown in Table. 1. Secondly, this research proposes a formation protocol that addresses each of the defined gaps, then assesses the proposed method in comparative simulations, in terms of maneuverability performance, tracking accuracy, constraint satisfaction, and estimation accuracy. Quantitative factors, including IAE, ITAE and RMSE, are employed in the assessment.

7. Conclusion

This paper addresses distributed formation control of underactuated AUVs moving in 3D space and subject to switching jointly connected topologies, unmeasurable velocities, and system constraints. A hybrid formation protocol that incorporates aspects of both centralized and decentralized control is proposed such that the objectives of distributed estimation and geometric convergence are achieved. The notable features of the proposed method can be summarized as follows. Firstly, the proposed method uses decentralized control in the sense that it assumes access to only local communication and local sensing, and therefore can be promoted to large-scale AUV swarms. Secondly, there is potential for wide real-world applicability, as the method does not have to rely
on velocity sensing, and can instead make use of any relative positioning technology [13]. Thirdly, limitations in practice, such as switching but occasionally disconnected topologies, external uncertainties, and system constraints, have been comprehensively considered in the proposed formation protocol. The comparative simulation results substantiate the effectiveness of the proposed method and its improvements over the state of the art.

Future work will still be required for practical deployments of large-scale AUV swarms, including collision avoidance and management of time delays. In addition, real experiments are still rare in multi-AUV research in general. It is of great importance that the community begin implementing state-of-the-art theoretical contributions on real AUVs to be verified in real environments.

Acknowledgment

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References


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Appendix A. Proof of Theorem 1

First, take note of the following two lemmas.

**Lemma 2 [49]:** A continuous positive definite function $V(t)$ is finite-time stable at the origin if there exist positive constants $\rho_1 > 0$, $\rho_2 > 0$, and $0 < c < 1$, such that

$$V(t) \leq -\rho_1 V(t) - \rho_2 V^c(t), \quad t \geq t_0,$$

(A.1)

and the settling time is given by

$$T_f \leq T_0 + \frac{1}{\rho_1 (1-c)} \ln \frac{\rho_1 V^{1-c}(T_0) + \rho_2}{\rho_2}.$$  

(A.2)

**Lemma 3 [50]:** Consider a group of systems

$$\dot{x}(t) = f_\sigma(x), f_\sigma(0) = 0, x \in \mathbb{R}^n$$

(A.3)

where $\sigma : [0, \infty) \to \{1, 2, \ldots, M\}$ is a piecewise constant function of time, $f_i$ is a continuous function with respect to $x$ for any fixed index $i \in \{1, 2, \ldots, M\}$. If the switched system is asymptotically stable and $\dot{x}(t) = f_i(x)$ is finite-time stable for any fixed $i$, then system Eq. (A.3) is finite-time stable.

**Proof:** For convenience, define $\mu_\eta = \begin{bmatrix} \mu_{\eta_1}^T, \mu_{\eta_2}^T, \ldots, \mu_{\eta_N}^T \end{bmatrix}^T$ and $\tilde{\eta}_0 = [\tilde{\eta}_{10}^T, \tilde{\eta}_{20}^T, \ldots, \tilde{\eta}_{N0}^T]^T$. It follows that

$$\mu_\eta(t) = P_\sigma(t) \tilde{\eta}_0(t).$$

(A.4)

where $P_\sigma(t) = (H_\sigma(t) \otimes I_0)$. Consider Lyapunov function $V_\eta = \mu_\eta^T \mu_\eta/2$, which is continuously differentiable at any non-switching instant. Inspired by Lemma 3, the proof will be presented in the following two steps.

**Step 1:** We will show that $\tilde{\eta}_0$ is asymptotically stable at the origin. Firstly, assume that graph $\tilde{G}_\sigma$ is active at non-switching time $t$. Differentiating $\mu_\eta$ and substituting Eq. (16) yields

$$\dot{\mu}_\eta = \begin{cases} -K_\eta \mu_\eta + \Phi_\eta - \kappa_1 \text{sgn} (\mu_\eta) - \kappa_2 \mu^c_\eta, & \alpha_i > 0 \\ 0, & \alpha_i = 0 \end{cases}.$$  

(A.5)

Note that $\alpha_i = 0$ is equivalent to the fact that $i \notin C(q)$ [36]. Therefore, taking the time derivative of $V_\eta$ yields

$$\dot{V}_\eta = \sum_{i \in C(q)} \mu_{\eta_i}^T \dot{\mu}_{\eta_i}$$

$$= \sum_{i \in C(q)} \mu_{\eta_i}^T \left( -K_\eta \mu_\eta + \Phi_\eta - \kappa_1 \text{sgn} (\mu_\eta) - \kappa_2 \mu^c_\eta \right)$$

$$= \sum_{i \in C(q)} -\mu_{\eta_i}^T K_\eta \mu_\eta + \mu_{\eta_i}^T \Phi_\eta - \kappa_1 |\mu_\eta| - \kappa_2 |\mu^c_\eta|^{1+c}$$

$$\leq \sum_{i \in C(q)} -\mu_{\eta_i}^T K_\eta \mu_\eta - (\kappa_1 - \gamma \Phi) |\mu_\eta| - \kappa_2 (\mu_{\eta_i}^T \mu_\eta)^{\epsilon + \frac{1}{2}}$$

$$\leq -\lambda_{\min}(K_\eta) \sum_{i \in C(q)} \mu_{\eta_i}^T \mu_\eta$$

$$\leq 0,$$  

(A.6)
where we use the fact that $|\mu_\eta|^2(\varsigma+1) \geq (\mu^T_\eta \mu_\eta)^{\varsigma+1}$ in the first inequality (based on Lemma 2 in [49]). Eq. (A.6) implies that \( \lim_{t \to \infty} V_\eta (t) \) exists.

Secondly, it is shown that \( \lim_{t \to \infty} \mu_\eta (t) = 0 \). Considering the infinite sequences \( V_\eta (t_k), k = 0, 1, \ldots \), and recalling Cauchy’s convergence criteria, it can be derived that for any \( \varepsilon > 0 \), there exists a positive integer \( k_\varepsilon \) such that, for \( \forall k > k_\varepsilon \),

$$
|V_\eta (t_{k+1}) - V_\eta (t_k)| < \varepsilon \quad \text{i.e.,} \quad \int_{t_k}^{t_{k+1}} -\dot{V}_\eta (t) \, dt < \varepsilon. \tag{A.7}
$$

Eq. (A.7) can be rewritten into the following sum of integrals:

$$
\sum_{l=0}^{l_\varepsilon} \int_{t_k}^{t_{k+1}} -\dot{V}_\eta (t) \, dt < \varepsilon. \tag{A.8}
$$

For each integral, following Eq. (A.6) yields that

$$
\int_{t_k}^{t_{k+1}} -\dot{V}_\eta (t) \, dt \geq \int_{t_k}^{t_{k+1}} \lambda_{\min} (K_\eta) \sum_{i \in C(\sigma(t_k^i))} \mu^T_{\eta i} \mu_\eta \, dt
\geq \lambda_{\min} (K_\eta) \int_{t_k}^{t_{k+1} + \tau} \sum_{i \in C(\sigma(t_k^i))} \mu^T_{\eta i} \mu_\eta \, dt.
\tag{A.9}
$$

Thus,

$$
\lambda_{\min} (K_\eta) \sum_{l=0}^{l_\varepsilon} \int_{t_k}^{t_{k+1} + \tau} \sum_{i \in C(\sigma(t_k^i))} \mu^T_{\eta i} \mu_\eta \, dt < \varepsilon, \tag{A.10}
$$

or, equivalently,

$$
\lim_{t \to \infty} \int_{t_k}^{t_{k+1} + \tau} \sum_{l=0}^{l_\varepsilon} \sum_{i \in C(\sigma(t_k^i))} \mu^T_{\eta i} (s) \mu_\eta (s) \, ds = 0. \tag{A.11}
$$

Due to the joint connectivity of the graph during \([t_k, t_{k+1}]\), Lemma 1 implies that

$$
\lim_{t \to \infty} \int_{t_k}^{t_{k+1} + \tau} \sum_{l=0}^{l_\varepsilon} a_i \mu^T_{\eta i} (s) \mu_\eta (s) \, ds = 0, \tag{A.12}
$$

where \( a_i \) \( (i = 1, \ldots, N) \) are some positive constants. Following Eq. (A.6), \( \mu_\eta (t) \) are uniformly bounded, and thus \( \mu_\eta (t) \) is bounded due to Eq. (A.5). Thus, \( \mu^T_{\eta i} \mu_\eta \) is uniformly continuous. Applying Barlalat’s Lemma, it can be derived that \( \lim_{t \to \infty} \mu_\eta (t) = 0 \).

Thirdly, it needs to be shown that \( \lim_{t \to \infty} \tilde{\eta}_0 (t) = 0 \). Without loss of generality, consider only the first element of \( \mu_\eta \) and \( \tilde{\eta}_0 \) in the proof, denoted by \( \mu^{\tilde{\eta}}_1 \) and \( \tilde{\eta}^{\tilde{\eta}}_0 \), respectively. Let \( \mu^{\tilde{\eta}}_1 = [\mu^{\tilde{\eta}}_{\eta 1}, \mu^{\tilde{\eta}}_{\eta 2}, \ldots, \mu^{\tilde{\eta}}_{\eta N}]^T \) and \( \tilde{\eta}^{\tilde{\eta}}_0 = [\tilde{\eta}^{\tilde{\eta}}_{\eta 0}, \tilde{\eta}^{\tilde{\eta}}_{\eta 20}, \ldots, \tilde{\eta}^{\tilde{\eta}}_{\eta N0}]^T \). Thus, \( \mu^{\tilde{\eta}}_1 = H_{\sigma(t)} \tilde{\eta}^{\tilde{\eta}}_0 \). Assume that the jointly connected graph across \([t_k, t_{k+1}]\) is denoted by \( \tilde{G}_k \). Due to the symmetry of \( H_k \), there exists an orthogonal matrix \( U_k \), such that

$$
U_k H_k U_k^T = \Lambda_k = \text{diag} \left\{ \lambda^1_k, \lambda^2_k, \ldots, \lambda^N_k \right\}, \tag{A.13}
$$

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where $\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_N}$ are the $N$ eigenvalues of $H_k$, $i_1, i_2, \ldots, i_N$ and form a permutation of $1, 2, \ldots, N$.

Let $\varepsilon = U_q\tilde{\eta}_0^T$, then
\[
(\tilde{\eta}_0^T)^T \mu_{\eta} = (\tilde{\eta}_0^T)^T H_k\tilde{\eta}_0^T = \varepsilon^T A_k \varepsilon \geq \delta_{\min} \sum_{i \in C(\sigma(t_k^i))} \varepsilon_i^2 \geq 0,
\]
where $l = 0, 1, \ldots, l_k - 1$ and $\delta_{\min} = \min \{\lambda_{\min}(H_q) | q \in Q\}$.

Following from the joint connectivity, Lemma 1, and $\lim_{t \to \infty} \mu_{\eta}^2(t) = 0$, it can be inferred that
\[
\lim_{t \to \infty} \sum_{i=1}^N \sum_{i \in C(\sigma(t_k^i))} \varepsilon_i^2 = \lim_{t \to \infty} \sum_{i=1}^N a_i \varepsilon_i^2 = 0.
\]
This implies that $\lim_{t \to \infty} \sum_{i=1}^N \varepsilon_i = 0$, i.e., $\lim_{t \to \infty} \tilde{\eta}_0^T = 0$. Thus, it can be concluded that $\lim_{t \to \infty} \eta_0(t) = 0$.

**Step 2:** We will show that, for a fixed index $q \in Q$, the estimation error $\tilde{\eta}_0$ is finite-time stable. Note that the switching topologies are jointly connected, and the $i$th follower cannot always be isolated. In other words, there exists at least one graph $\tilde{G}_q'$ with $q' \in Q$ such that $\alpha_i > 0$. Thus, we consider the Lyapunov function candidate $V_{\eta_i} = \mu_{\eta_i}^T \mu_{\eta_i}/2$ under graph $\tilde{G}_q'$. Following Eq. (A.6), the derivative is given by
\[
\dot{V}_{\eta_i} = \sum_{i \in C(q)} \mu_{\eta_i}^T \left( -K_{\eta_i} \mu_{\eta_i} + \Phi_{\eta_i} - \kappa_1 \text{sgn} (\mu_{\eta_i}) - \kappa_2 \text{sgn}^2 (\mu_{\eta_i}) \right)
\]
\[
\leq -\mu_{\eta_i}^T K_{\eta_i} \mu_{\eta_i} - (\kappa_1 - \gamma_0) |\mu_{\eta_i}| - \kappa_2 (\mu_{\eta_i}^T \mu_{\eta_i})^{\frac{\gamma_1}{2}}
\]
\[
\leq -\lambda_{\min}(K_{\eta}) V_{\eta_i} - \kappa_2 2^{\frac{\gamma_1}{2}} V_{\eta_i}^{\frac{\gamma_1}{2}}.
\]
It follows from Lemma 2 that $\mu_{\eta_i}$ converges to zero in finite time $T_{f_i}$, bounded by
\[
T_{f_i} \leq T_0 + \frac{2}{\lambda_{\min}(K_{\eta}) (1 - \gamma_0)} \ln \left( \frac{\lambda_{\min}(K_{\eta}) 2^{-\frac{\gamma_1}{2}} V_{\eta_i}^{\frac{\gamma_1}{2}} (T_0) + \kappa_2}{\kappa_2} \right).
\]
The remaining proof is similar to that of Step 1. According to Lemma 3, we can conclude that the estimation error $\tilde{\eta}_0$ converges to zero in finite time.

---

**Appendix B. Proof of Theorem 2**

**Proof:** Given the cascaded form of the proposed ESOs, the proof is presented in three steps, as follows.

**Step 1:** Let $\pi_{qi} = [\tilde{\theta}_i, \tilde{q}_i, \tilde{D}_{qi}]^T$ such that $\tilde{\theta}_i = \hat{\theta}_i - \theta_i$, $\tilde{q}_i = \hat{q}_i - q_i$, and $\tilde{D}_{qi} = \hat{D}_{qi} - D_{qi}$. It follows from Eq. (1), Eq. (3), and Eq. (17) that
\[
\pi_{qi} = \begin{bmatrix}
-k_{q_1} & 1 & 0 \\
-k_{q_2} & 0 & 1 \\
-k_{q_3} & 0 & 0 \\
\end{bmatrix}
\pi_{qi} + \begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix},
\]
\[
\pi_{qi} = \begin{bmatrix}
-l_{q_1} & 0 & 0 \\
-l_{q_2} & 0 & 0 \\
-l_{q_3} & 0 & 0 \\
\end{bmatrix}
\pi_{qi} + \begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}.
\]
The stability of the error dynamics system (B.1) can be guaranteed if there exist a positive definite matrix $Q_q \in \mathbb{R}^{3 \times 3}$ and a positive constant $\epsilon_q > 1$ such that

$$L_q^T Q_q + Q_q L_q \leq -\epsilon_q I_3. \tag{B.2}$$

Consider the Lyapunov function $V_{qi} = \pi_{qi}^T Q_{qi} \pi_{qi}$, with its derivative as

$$\dot{V}_{qi} = \pi_{qi}^T (L_q^T Q_q + Q_q L_q) \pi_{qi} + 2 \pi_{qi}^T Q_q N_{qi}$$
\[\leq -\epsilon_q \pi_{qi}^T \pi_{qi} + 2 \|Q_q\| \|N_{qi}\|\]
\[\leq -\|\pi_{qi}\| (\epsilon_q \|\pi_{qi}\| - 2 \|Q_q\| N_q). \tag{B.3}\]

It can then be derived from Eq. (B.3) that $\|\pi_{qi}\| \leq 2 \|Q_q\| \tilde{N}_q/\epsilon_q$ and that the upper bound can be decreased by choosing appropriate parameters.

**Step 2:** Define $\pi_{ri} = \begin{bmatrix} \tilde{\xi}_{ri}, \tilde{r}_i, \tilde{D}_{ri} \end{bmatrix}^T$, where $\tilde{\xi}_{ri} = \dot{\xi}_{ri} - \xi_{ri}, \tilde{r}_i = \dot{r}_i - r_i$, and $\tilde{D}_{ri} = \dot{D}_{ri} - D_{ri}$. It can then be deduced that

$$\dot{\pi}_{ri} = \begin{bmatrix} -k_{r1} & 1 & 0 \\ -k_{r2} & 0 & 1 \\ -k_{r3} & 0 & 0 \end{bmatrix} \pi_{qi} + \begin{bmatrix} -\tilde{q}_i \psi_1 \sin \theta_i \\ 0 \\ -\tilde{D}_{ri} \end{bmatrix}. \tag{B.4}$$

Step 1 and Assumption 2 guarantee the boundedness of $\|N_{ri}\|$. Analogously to Step 1, $\|\pi_{ri}\|$ will also be bounded by choosing appropriate parameters appropriately.

**Step 3:** Define $\pi_{pi} = \begin{bmatrix} \xi_{pi}, \tilde{\nu}_{B_i}, \tilde{D}_{pi} \end{bmatrix}^T$, where $\tilde{\pi}_{pi} = \dot{\pi}_{pi} - \pi_{pi}, \tilde{\nu}_{B_i} = \dot{\nu}_{B_i} - \nu_{B_i}$, and $\tilde{D}_{pi} = \dot{D}_{pi} - D_{pi}$. The following can then be obtained:

$$\dot{\pi}_{pi} = \begin{bmatrix} -K_{p1} & I_3 & 0 \\ -K_{p2} & 0 & I_3 \\ -K_{p3} & 0 & 0 \end{bmatrix} \pi_{qi} + \begin{bmatrix} g_{pi} \\ 0 \\ -\tilde{D}_{pi} \end{bmatrix}, \tag{B.5}$$

where $g_{pi} = S_{I_i}^{B_i}(\dot{\hat{r}}_i, \dot{\hat{q}}_i, \theta_i) \xi_{pi} - S_{I_i}^{B_i}(r_i, q_i, \theta_i) \xi_{pi} - R_{I_i}^{B_i}(\dot{p}_0^e - \dot{p}_0)$. From Theorem 1, it follows that $\lim_{t \to \infty} \xi_{pi}(t) = \xi_{pi}(t) = 0$ and $\lim_{t \to \infty} \dot{p}_0^e(t) = \dot{p}_0(t) = 0$. The ultimate boundedness of $S_{I_i}^{B_i}(\dot{\hat{r}}_i, \dot{\hat{q}}_i, \theta_i) - S_{I_i}^{B_i}(r_i, q_i, \theta_i)$ can then also be proven, according to the former steps. It follows that there exists a positive constant $\tilde{N}_p$ such that $\|N_{pi}\| \leq \tilde{N}_p$. The remainder of the proof for Step 3 is omitted here, as it can be inferred from Step 1.

**Appendix C. Proof of Theorem 3**

**Proof:** From Eq. (17), the dynamic model of $\dot{q}_i$ can be expressed as

$$\dot{q}_i = -k_{q2} \dot{\theta}_i + (\tau_{qi} + \Delta \tau_{qi})/m_{qi} + \dot{D}_{qi}. \tag{C.1}$$

Then, differentiating $s_{qi}$ while introducing Eq. (33) and Eq. (C.1) yields

$$\dot{s}_{qi} = -c_q \dot{s}_{qi} + c_q \phi_{\tau_{qi}} - k_{q2} \dot{\theta}_i + \Delta \tau_{qi}/m_{qi}. \tag{C.2}$$
Consider the Lyapunov candidate
\[ V_{qi} = \frac{s_{qi}^2}{2} + \phi_{qi}^2/2 + \phi_{\tau qi}^2/2. \]  
(C.3)
Differentiating \( V_{ui} \) while substituting Eq. (C.2), Eq. (34), and Eq. (35) then yields
\[ \dot{V}_{qi} = s_{qi} \left( -c_q s_{qi} + c_q \phi_{\tau ui} - k_q \bar{\theta}_i + \Delta \tau_{qi} / m_{qi} \right) 
+ \phi_{qi} \left( -w_1 \phi_{qi} - \frac{w_2 \Delta q_i^2}{2 |\phi_{qi}|} \phi_{qi} + w_2 \Delta \phi_i \right) 
+ \phi_{\tau qi} \left( -l_1 \phi_{\tau qi} - |g_{\tau qi}| + \frac{l_2 \Delta \tau_{qi}^2}{2} \phi_{\tau qi} + l_2 \Delta \phi_{\tau qi} \right) \]  
(C.4)
\[ = -c_q s_{qi}^2 + c_q s_{qi} \phi_{\tau qi} - k_q \phi_{\tau qi} \bar{\theta}_i + g_{\tau qi} 
- w_1 \phi_{qi}^2 - w_2 \Delta q_i^2 / 2 + w_2 \phi_{\tau qi} \Delta \phi_i 
- l_1 \phi_{\tau qi}^2 - |g_{\tau qi}| - l_2 \Delta \tau_{qi}^2 / 2 + l_2 \phi_{\tau qi} \Delta \tau_{qi}. \]

Recall the existence of the inequalities
\[
\begin{cases} 
2s_q \phi_{\tau qi} \leq s_{qi}^2 + \phi_{\tau qi}^2 \\
2\phi_{qi} \Delta \phi_i \leq \phi_{qi}^2 + \Delta q_i^2 \\
2\phi_{\tau qi} \Delta \tau_{qi} \leq \phi_{\tau qi}^2 + \Delta \tau_{qi}^2 
\end{cases}
\]  
(C.5)
It then follows that Eq. (C.4) can be reorganized as
\[ \dot{V}_{qi} \leq -c_q e_{ui}^2 / 2 - (2w_1 - w_2) \phi_{ui}^2 / 2 
- (2l_1 - c_q - l_2) \phi_{\tau ui}^2 / 2 - k_q \phi_{ui} \bar{\theta}_i, \]  
where the parameters are chosen such that \( c_q > 0, 2w_1 > w_2 \) and \( 2l_1 > c_q + l_2 \). After defining \( E_{qi} = [s_{qi}, \phi_{qi}, \phi_{\tau qi}]^T \), Eq. (C.6) can then be rewritten as
\[ \dot{V}_{qi} \leq -E_{qi}^T K_q E_{qi} + E_{\alpha i}^T \bar{T}_{qi} \]  
\[ \leq -\|E_{qi}\| (\|K_q\| \|E_{qi}\| - \|T_{qi}\|), \]  
(C.7)
where \( K_q = diag \left( c_q / 2, w_1 - w_2 / 2, l_1 - (c_q + l_2) / 2 \right) \) and \( T_{ui} = \left[ -k_q \bar{\theta}_i, 0, 0 \right]^T \). Theorem 2 implies that \( \|T_{qi}\| \leq \bar{T}_{qi} \), with \( \bar{T}_{qi} \) being a positive constant. It can therefore be concluded that \( \|E_{qi}\| \leq \bar{T}_{qi} / \lambda_{\min} (K_q) \), and the upper bound can be decreased by adjusting the parameters.

Appendix D. Proof of Theorem 4

Proof: The proof consists of two steps, as follows.

Step 1: After defining angle tracking error \( e_{ai} = [e_{\theta i}, e_{\psi i}]^T = [\theta_i - \theta_i^d, \psi_i - \psi_i^d]^T \), it can be proven that \( e_{ai} \) is ultimately bounded. When considering Lyapunov function candidate \( V_{ai} = e_{ai}^T e_{ai} / 2 \), differentiating \( V_{ai} \) with respect to time yields
\[ \dot{V}_{ai} = e_{\theta i} \left( q_i - \dot{\theta}_i \right) + e_{\psi i} \left( r_i / \cos \theta_i - \dot{\psi}_i \right). \]  
(D.1)
Recalling that $e_{qi} = q_i - q_i^d$ and $e_{ri} = r_i - r_i^d$, then substituting Eq. (29) into Eq. (D.1) yields that
\[
\dot{V}_{ai} = e_{qi}^T (-b_\psi e_{\theta i} + e_{qi}) + e_{\psi i}^T (-b_\psi e_{\psi i} + e_{ri}/\cos \theta_i)
= -\mathbf{e}_{ai}^T \begin{bmatrix} b_\psi & 0 \\ 0 & b_\psi \end{bmatrix} \mathbf{e}_{ai} + \mathbf{e}_{ai}^T \begin{bmatrix} e_{qi} \\ e_{ri}/\cos \theta_i \end{bmatrix} \mathbf{M}_{ai}
\leq -\|\mathbf{e}_{ai}\| (\|\mathbf{L}_a\| \|\mathbf{e}_{ai}\| - \|\mathbf{M}_{ai}\|).
\]

Under Theorem 3, it can be deduced that $\|\mathbf{M}_{ai}\|$ is bounded by a positive constant denoted by $M_{ai}$. It therefore follows that $\|\mathbf{e}_{ai}\| \leq M_{ai}/\lambda_{\min} (\mathbf{L}_a)$.

**Step 2:** The ultimate boundedness of $\mathbf{e}_{pi}$ is given as follows. Consider Lyapunov function candidate $V_{pi} = \mathbf{e}_{pi}^T \mathbf{e}_{pi}/2$ and differentiating $V_{pi}$ along Eq. (12), it yields that
\[
V_{pi} = e_{xi} (U_i \cos \psi_{ei} \cos \theta_{ei} - U_0 + f_{xi})
+ e_{yi} (U_i \sin \psi_{ei} \cos \theta_{ei} + f_{yi})
+ e_{zi} (-U_i \sin \theta_{ei} + f_{zi}).
\]

Then, note that
\[
\mathbf{R}_{B_i}^F (\psi_{ei}, \theta_{ei}) - \mathbf{R}_{B_i}^d (\psi_{ei}, \theta_{ei})
= (\mathbf{R}_F^I (\chi_0, \nu_0))^T \mathbf{R}_{B_i}^I (\psi_{ei}, \theta_{ei}) - (\mathbf{R}_F^I (\hat{\chi}_0, \hat{\nu}_0))^T \mathbf{R}_{B_i}^I (\psi_{ei}^d, \theta_{ei}^d)
= (\mathbf{R}_F^I (\chi_0, \nu_0))^T \left[ \mathbf{R}_{B_i}^I (\psi_{ei}, \theta_{ei}) - \mathbf{R}_{B_i}^I (\psi_{ei}^d, \theta_{ei}^d) \right]
+ (\mathbf{R}_F^I (\chi_0, \nu_0) - \mathbf{R}_F^I (\hat{\chi}_0, \hat{\nu}_0))^T \mathbf{R}_{B_i}^I (\psi_{ei}^d, \theta_{ei}^d).
\]

Under Theorem 1 and Step 1, it can be shown that $\mathbf{R}_{B_i}^F (\psi_{ei}, \theta_{ei}) - \mathbf{R}_{B_i}^d (\psi_{ei}, \theta_{ei})$ converge to a small neighborhood of zero. Then, expanding Eq. (D.4), one can obtain that
\[
\begin{align*}
\cos \psi_{ei} \cos \theta_{ei} - \cos \psi_{ei}^d \cos \theta_{ei}^d & \leq e_{Rx} \\
\sin \psi_{ei} \cos \theta_{ei} - \sin \psi_{ei}^d \cos \theta_{ei}^d & \leq e_{Ry}, \\
\sin \theta_{ei} - \sin \theta_{ei}^d & \leq e_{Rz}
\end{align*}
\]

where $e_{Rx}$, $e_{Ry}$, and $e_{Rz}$ are positive constants. Substituting Eq. (D.5) into Eq. (D.3) then yields
\[
\dot{V}_{pi} \leq e_{xi} (U_i \cos \psi_{ei}^d \cos \theta_{ei}^d - U_0 + f_{xi} + U_i e_{Rx})
+ e_{yi} (U_i \sin \psi_{ei}^d + f_{yi} + U_i (e_{Ry} + \theta_{ei}^d h (\psi_{ei}^d, \theta_{ei}^d)))
+ e_{zi} (-U_i \sin \theta_{ei}^d + f_{zi} + U_i e_{Rz}).
\]
where \( h (\psi_{ci}^d, \theta_{ci}^d) = \sin \psi_{ci}^d (\cos \theta_{ci}^d - 1) / \theta_{ci}^d \) is an auxiliary function. Given that \(|(\cos \theta_{ci}^d - 1) / \theta_{ci}^d| < 0.73[51]\), it is guaranteed that \(|h (\psi_{ci}^d, \theta_{ci}^d)| < 1\). Introducing Eq. (24) into Eq. (D.6) then yields

\[
\dot{V}_{pi} \leq e_{xi} \left( \frac{U_i}{\Delta_i} - U_0 + f_{xi} + U_i e_{Rx} \right) + e_{yi} \left( \frac{f_{yi} + U_i \left( e_{Ry} + \left| \theta_{ci}^d \right| \right)}{\Delta_z} \right) + e_{zi} \left( \frac{f_{zi} + U_i e_{Rz}}{\Delta_i \Delta_y} \right).
\]

Based on Eq. (25), it follows that

\[
\left\{ \begin{array}{l}
\dot{U}_i \delta_{yi} / \Delta_z = \hat{f}_{yi} \\
\dot{U}_i \delta_{zi} / \Delta_i \Delta_y = \hat{f}_{zi}
\end{array} \right.
\]

Under Theorem 1 and Theorem 2, it can be deduced that \( \hat{U}_i = \dot{U}_i - U_i, \hat{f}_{zi} = \dot{f}_{zi} - f_{zi}, \hat{f}_{yi} = \dot{f}_{yi} - f_{yi}, \) and \( \hat{f}_{zi} = f_{zi} - f_{zi} \) are ultimately bounded. It can be further proven that

\[
\left\{ \begin{array}{l}
\hat{f}_{zi} \leq e_{fx} \\
|\hat{f}_{zi}| \leq e_{fy},
\end{array} \right.
\]

where \( e_{fx}, e_{fy}, \) and \( e_{fx} \) are positive constants. By referring to Eq. (5), one has

\[
U_i = \frac{u_i}{\cos \alpha_i \cos \beta_i} = \frac{u_i^d + e_{ui}}{\cos \alpha_i \cos \beta_i}.
\]

Introducing Eq. (D.9) and Eq. (D.10) into Eq. (D.7) then yields that

\[
\dot{V}_{pi} \leq e_{xi} \left( -b_u e_{xi} - b_u \hat{e}_{xi} + \hat{U}_{i0} + e_{fx} + U_i e_{Rx} + f_{ui} \right) + e_{yi} \left( \frac{e_{yi}}{\Delta_z} + e_{fy} + U_i \left( e_{Ry} + \left| \theta_{ci}^d \right| - \hat{e}_{yi} \right) \right) + e_{zi} \left( \frac{e_{zi}}{\Delta_i \Delta_y} + e_{fz} + U_i \left( e_{Rz} - \hat{e}_{zi} \Delta_z \right) \right)
\]

\[
\leq -e_{pi}^T \begin{bmatrix} b_u & 0 & 0 \\ 0 & U_i & 0 \\ 0 & 0 & \hat{U}_{i0} \end{bmatrix} e_{pi} + e_{pi}^T \begin{bmatrix} -b_u \hat{e}_{xi} + \hat{U}_{i0} + e_{fx} + U_i e_{Rx} + f_{ui} \\ e_{fy} + U_i \left( e_{Ry} + \left| \theta_{ci}^d \right| - \hat{e}_{yi} \right) \right. \\
\left. \left. e_{fz} + U_i \left( e_{Rz} - \hat{e}_{zi} \Delta_z \right) \right) \right) \leq -e_{pi}^T L_{pi} e_{pi} + e_{pi}^T M_{pi} e_{pi}
\]

\[
36
\]
where \( \tilde{e}_{xi} = e_{xi} - e_{xi}, \tilde{e}_{yi} = e_{yi} - e_{yi}, \tilde{e}_{zi} = e_{zi} - e_{zi}, \tilde{U}_{i0} = \tilde{U}_{i0} - U_{i0} \) and \( f_{ui} = e_{ui}/(\cos \alpha_i \cos \beta_i \Delta_i) \).

Note that \( \tilde{e}_{xi}, \tilde{e}_{yi}, \tilde{e}_{zi}, \tilde{U}_{i0} \) are estimation errors induced by the DO, which are guaranteed to be bounded and converge to zero in finite time, \( e_{fx}, e_{fy}, e_{fz} \) are estimation errors induced by the CESO, and \( e_{Rx}, e_{Ry}, e_{Rz}, f_{ui} \) are tracking errors induced by the dynamic controller. According to Theorems 1–3, these error signals all converge to a small neighborhood around zero. In other words, \( \| M_{pi} \| \) is lower and upper bounded by a positive constant and \( M_{pi} \), respectively, and the upper bound \( M_{pi} \) can be minimized by adjusting controller parameters according to the guidelines given in Appendix E. Consequently, it can be concluded that \( \| e_{pi} \| \leq M_{pi}/\lambda_{\min}(L_{pi}) \).

**Appendix E. A discussion of control parameters**

It is well known that the tuning of control parameters is not a trivial task, either in simulation or reality. Therefore, some design guidelines for the observer and control parameters are provided in this appendix, which can also help readers compromise between tracking accuracy, transient response, saturation avoidance, and estimation performance.

**Appendix E.1. Observer parameters**

For the proposed DO Eq. (16), the observer gain \( K_{\eta} \) should be a diagonal positive definite matrix. On one hand, the increase of \( K_{\eta} \) can speed up the estimation process, as shown in Eq. A.6, but large values might result in instability. On the other hand, small values can slow down the estimation process. In addition, it is required that \( \kappa_1 > \gamma_\varphi, \kappa_2 > 0 \) and \( \varsigma \) satisfies \( 0 < \varsigma = \varsigma_1/\varsigma_2 < 1 \) with \( \varsigma_1 \) and \( \varsigma_2 \) being positive odd integers. The increase of \( \kappa_1 \) will amplify noncontinuous signal \( \text{sign}(\mu_{pi}) \), which might lead to chattering phenomena. From Eq. A.17, we can observe that large value of \( \kappa_2 \) can reduce the settle time.

In summary, \( K_{\eta} \) should be increased gradually to achieve the desired estimation response, and \( \beta_\eta \) can be tuned according to the value of \( \gamma_\varphi \).

For the proposed CESO Eqs. (17, 18, and 21), the gains can be chosen as

\[
\begin{align*}
\kappa_{q1} &= 3\omega_q, \kappa_{r1} = 3\omega_r, K_{p1} = 3\omega_p I_3, \\
\kappa_{q2} &= 3\omega_q^2, \kappa_{r2} = 3\omega_r^2, K_{p2} = 3\omega_p^2 I_3, \\
\kappa_{q3} &= \omega_q^3, \kappa_{r3} = \omega_r^3, K_{p3} = \omega_p^3 I_3,
\end{align*}
\]

(E.1)

where \( \omega_q, \omega_r, \) and \( \omega_p \) are observer bandwidths. By referring to the proof of Theorem 2, it can be noticed that the estimation error can be tuned to be arbitrarily small by increasing the observer bandwidths. However, the bandwidths are normally limited due to measurement noise. This is an unavoidable shortcoming of ESOs, as discussed in [52]. Therefore, \( \omega_q, \omega_r, \) and \( \omega_p \) should be tuned by taking into consideration both sensitivity to noise and the estimation performance.

**Appendix E.2. Control parameters**

Regarding the LOS guidance law Eq. (24), and according to Eq. (25), the decrease of \( \Delta_y \) can reduce the response time and minimize the tracking error. However, small \( \Delta_y \) values will also lead to large velocity command signals, which may violate the system constraints. Consequently, \( \Delta_y \) should be determined by compromising between tracking accuracy and saturation.

For kinematic controller Eq. (29), larger values of gains \( b_u, b_q, \) and \( b_p \) guarantee better tracking accuracy, but system constraints might also be violated as gains increase. A trade off between tracking accuracy and saturation should therefore also be considered when tuning these gains.
For dynamic controller Eq. (33), \( c_q > 0 \), \( 2w_1 > w_2 \), and \( 2l_1 > c_q + l_2 \) should first be satisfied. The increase of \( c_q \) can improve the velocity tracking accuracy, but might also increase the control input amplitude, leading to input saturation. An increase of \( w_2 \) and \( l_2 \) can enhance controller sensibility against saturation, and an increase of \( w_1 \) and \( w_2 \) can improve robustness against saturation. However, for the sake of stability, the gains cannot be too large.