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Some Combinatorial Optimization Problems on which Ant Colony Optimization is Invariant

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Abstract

Ant colony optimization is a well known metaheuristic which has been object of many studies, both theoretic and applicative. In a recent analysis [1], a particular aspect of its behavior is investigated. More in detail, the topic object of interest is the result achieved by ant algorithms when the cost unit in which instances are expressed changes. Three ant colony optimization algorithms are proved to be invariant to this type of transformation of instances, provided that some conditions are satisfied. In this paper some examples of combinatorial optimization problems to which ant colony optimization can be applied in an invariant fashion are described. Many of the problems typically tackled in the literature may be included in this list.

Introduction

Ant colony optimization (ACO) [2] is a well known metaheuristic, which is used for tackling many combinatorial optimization problems. Recently, the issue of the invariance of ACO algorithm to the scaling of instances has been risen, and a proof of this property has been proposed [1]. Typically, the goal in a combinatorial optimization problem is the minimization of a function based on quantities which often represent time intervals, distances or, more in general, costs. Informally, an algorithm is said to be invariant to the scaling of instances if the sequence of solutions generated remains unchanged when tackling two instances that differ only for the cost unit adopted. For a formal definition of the problem we refer the reader to Birattari et al. [1].

In the following, we indicate with I and $\bar{I}$ two equivalent instances ($\bar{I} = g_1 I$), i.e. two instances that are related via a linear transformation of units [1]. They are such if they share the same space of solutions S and, for any solution $s \in S$, $\bar{f}(s) = g_1 f(s)$, where $g_1 > 0$ is a constant and $f(s)$ and $\bar{f}(s)$ are the value of the objective function in s for I and $\bar{I}$, respectively.

In the notation adopted, if y is a generic quantity that refers to an instance I, then $\bar{y}$ is the corresponding quantity for what concerns instance $\bar{I}$.

Moreover, let s_0 be a reference solution of instance I, returned by some appropriate invariant algorithm. Such an algorithm, which is necessarily problem-specific, might be based either on a heuristic or more simply on a random sampling of the solution space. From this definition, it follows that $\bar{f}(s_0) = g_1 f(s_0)$, for any two equivalent instances I and $\bar{I}$ such that $\bar{I} = g_1 I$.

Finally, in ant colony optimization, a combinatorial optimization problem is mapped on a graph $G = (N, E)$, where N is the set of nodes and E is the set of edges. The graph G is called *construction graph*. The solutions to the original problem are mapped to paths on G. Variables called *pheromone* and *heuristic information* are associated with the edges in E. In the following, we will adopt the notation $\langle i, j\rangle$ to denote the edge connecting nodes i and j. With η_{ij} we will denote the heuristic information on the desirability of constructing a path on G featuring node j immediately after i.

In Birattari et al. [1], the algorithmic framework of ant system, ant colony system, and $\mathcal{MAX}$–$\mathcal{MIN}$ ant system is described. These three algorithms are shown to be invariant to the scaling of instances when the following condition is verified:

(Condition 1) the heuristic information is such that

$$[\bar{\eta}_{ij}]^\beta = [g_2\eta_{ij}]^\beta, \text{ for all } \langle i, j\rangle \in E,$$

where β is a parameter and $g_2 > 0$ is an arbitrary constant.

Moreover, the paper introduces three algorithms: *si*AS, *si*$\mathcal{MM}$AS, and *si*ACS. These algorithms are *functionally equivalent* to ant system, $\mathcal{MAX}$–$\mathcal{MIN}$ ant system, and ant colony system, respectively, but they enjoy the further property of being *strongly invariant*. The first property ensures that two algorithms generate the same sequence of solutions for any instance I. The second property ensures that, beside producing the same sequence of solutions irrespectively of any linear rescaling of the problem instance, the algorithms are such that the *pheromone* and the *heuristic information* do not change with the scale of the problem instance. The only problem-specific element which must be suitably defined for obtaining the strongly invariant version of the algorithms is the heuristic information. In particular, Conditions 2 and 3 are to be met for achieving the functional equivalence and the strong invariance, respectively.

(Condition 2) the heuristic information is such that

$$[\tilde{\eta}_{ij}]^\beta = [\lambda\eta_{ij}]^\beta, \text{ for all } \langle i, j\rangle,$$

where β is a parameter, $\tilde{\eta}_{ij}$ and η_{ij} are the heuristic information on edge $\langle i, j\rangle$ respectively in the strongly invariant algorithm and in the classic one, and $\lambda > 0$ is an arbitrary constant.

(Condition 3) the heuristic information used in the strongly invariant algorithm is such that

$$[\bar{\eta}_{ij}]^\beta = [\eta_{ij}]^\beta, \text{ for all } \langle i, j\rangle,$$

for any two instances I and $\bar{I}$ such that $\bar{I} = g_1 I$, with $g_1 > 0$.

In the following, some examples of typical problems to which ACO can be applied in an invariant fashion are described. A setup for having the three conditions satisfied is pointed out. As it can be observed, many of the typical combinatorial optimization problems can be tackled with ant algorithms obtaining results that are not dependent on the scale of the cost unit used.

1 Traveling salesman problem

The traveling salesman problem (TSP) consists in finding a Hamiltonian circuit of minimum cost on an edge-weighted graph $G = (N, E)$. Let N be the set of nodes, and E be the set of edges. If a directed graph is considered, the problem is known as the *asymmetric* traveling salesman problem [3].

Let $x_{ij}(s)$ be a binary variable taking value 1 if edge $\langle i, j\rangle$ is included in tour s, and 0 otherwise. Let c_{ij} be the cost associated to edge $\langle i, j\rangle$. The goal is to find a tour s such that the function

$$f(s) = \sum_{i\in N}\sum_{j\in N} c_{ij}x_{ij}(s)$$

is minimized.

1) Transformation of units: If the cost of all edges is multiplied by a constant ζ, the resulting instance $\bar{I}$ is equivalent to the original I, that is, $\bar{I} = g_1 I$, with $g_1 = \zeta$. Indeed, $\bar{c}_{ij} = \zeta c_{ij}$, for all $\langle i, j \rangle \implies \bar{f}(s) = \zeta f(s)$, for all s.

2) Reference solution: Many constructive heuristics exist for the TSP [4] that can be conveniently adopted here.

3) Heuristic information: The typical setting is $\eta_{ij} = 1/c_{ij}$, for all $\langle i, j \rangle$. This meets Condition 1 with $g_2 = 1/\zeta$.

Therefore, the theorems on the weak invariance of ant system, $\mathcal{MAX}$–$\mathcal{MIN}$ ant system, and ant colony system hold. In the literature, the three variants of ant colony optimization considered in this paper have been applied to the traveling salesman problem with the setup just described [2].

4) Strongly-invariant heuristic information: Let $\eta_{ij} = f(s_0)/nc_{ij}$, for all $\langle i, j \rangle$, where $n = |N|$. It is worth noting that the term n is not needed for the invariance to transformation of units. It has been included for achieving another property: the above defined η_{ij} does not depend on the size of the instance under analysis—that is, on the number n of cities. This definition meets Condition 2 with $\lambda = f(s_0)/n$, and Condition 3.

Therefore, *si*AS, *si*$\mathcal{MM}$AS, and *si*ACS are indeed strongly invariant and are functionally equivalent to their original counterparts.

2 Vehicle routing problem

In the vehicle routing problem (VRP) n customers have to be served starting from one central depot (which is indexed 0). Each customer i has a non-negative demand d_i of the same good, and for each pair of customers $\langle i, j \rangle$ a travel time c_{ij} between the two customers is given. The customers are served by a fleet of vehicles of equal capacity. The objective is to find a set of routes that minimizes the total travel time, such that: each customer is served once by exactly one vehicle; the route of each vehicle starts and ends at the depot; and the total demand covered by each vehicle does not exceed its capacity. Using the notation introduced for the traveling salesman problem, the objective function can be written as:

$$f(s) = \sum_{i \in N} \sum_{j \in N} c_{ij} x_{ij}(s).$$

Variants of the problem may have different goals, as for example the minimization of the number of vehicles used [5].

1) Transformation of units: If the cost of all edges is multiplied by a constant ζ, the resulting instance $\bar{I}$ is equivalent to the original I, that is, $\bar{I} = g_1 I$, with $g_1 = \zeta$. Indeed, $\bar{c}_{ij} = \zeta c_{ij}$, for all $\langle i, j \rangle \implies \bar{f}(s) = \zeta f(s)$, for all s.

2) Reference solution: Many constructive heuristics exist for the VRP [5] that can be conveniently adopted here.

3) Heuristic information: Let $\eta_{ij} = \int_{ij}$ [6], with $\int_{ij}$ measure of the cost saving that would be achievable by using edge $\langle i, j \rangle$ [7]: We first consider a solution with one separate tour per customer, i.e. tours starting and ending at the depot and touching only one customer each. Then, for each pair $\langle i, j \rangle$ of customers, a saving $\int_{ij} = c_{i0} + c_{0j} - c_{ij}$ is computed. This meets Condition 1 with $g_2 = \zeta$.

Therefore, the theorems on the weak invariance of ant system, $\mathcal{MAX}$–$\mathcal{MIN}$ ant system, and ant colony system hold. In the literature, variants of the VRP have been tacked with ACO algorithm. Limiting this survey to the standard version, let us cite, among the others, Reimann et al. [8], Bullnheimer et al. [6].

4) Strongly-invariant heuristic information: Let $\eta_{ij} = n\int_{ij}/f(s_0)$, for all $\langle i,j\rangle$, where $n = |N|$. As in the case of the traveling salesman problem, the term n is not needed for the invariance to transformation of units. It has been included for achieving another property: the above defined η_{ij} does not depend on the size of the instance under analysis—that is, on the number n of cities. This definition meets Condition 2 with $\lambda = n/f(s_0)$, and Condition 3.

Therefore, *si*AS, *si*$\mathcal{MM}$AS, and *si*ACS are indeed strongly invariant and are functionally equivalent to their original counterparts.

3 Sequential ordering problem

The sequential ordering problem (SOP) consists in finding a minimum weight hamiltonian path on a directed graph with weights on edges and nodes, subject to precedence constraints [9]. It is often tackled as a constrained version of the asymmetric traveling salesman problem, by associating to each edge $\langle i,j\rangle$ a cost c'_{ij}:

$$c'_{ij} = \begin{cases} \infty & \text{if node } j \text{ has to precede node } i, \\ c_{ij} + p_j & \text{otherwise,} \end{cases}$$

where c_{ij} is the weight of edge $\langle i,j\rangle$, and p_j is the weight of node j. With this feature the objective function can be expressed as

$$f(s) = \sum_{i\in N}\sum_{j\in N} c'_{ij}x_{ij}(s)$$

using the notation introduced for the TSP.

1) Transformation of units: If the cost of both edges and nodes is multiplied by a constant ζ, then the cost c'_{ij} is scaled by the same constant ζ, for all $\langle i,j\rangle$. The resulting instance $\bar{I}$ is equivalent to the original I, that is, $\bar{I} = g_1 I$, with $g_1 = \zeta$. Indeed, $\bar{c}'_{ij} = \zeta c'_{ij}$, for all $\langle i,j\rangle \implies \bar{f}(s) = \zeta f(s)$, for all s.

2) Reference solution: For the construction of a reference solution, one of the heuristics available for the TSP [4] can be conveniently adopted.

3) Heuristic information: The typical setting is $\eta_{ij} = 1/c'_{ij}$, for all $\langle i,j\rangle$. This meets Condition 1 with $g_2 = 1/\zeta$.

Therefore, the theorems on the weak invariance of ant system, $\mathcal{MAX}$-$\mathcal{MIN}$ ant system, and ant colony system hold. One of the best algorithms available in the literature for the sequential ordering problem is the hybrid ant colony system proposed by Gambardella and Dorigo [10], where the authors use this invariant setup.

4) Strongly-invariant heuristic information: Let $\eta_{ij} = f(s_0)/nc'_{ij}$, for all $\langle i,j\rangle$, where $n = |N|$. This definition meets Condition 2 with $\lambda = f(s_0)/n$, and Condition 3.

Therefore, *si*AS, *si*$\mathcal{MM}$AS, and *si*ACS are indeed strongly invariant and are functionally equivalent to their original counterparts.

4 Quadratic assignment problem

In the quadratic assignment problem (QAP), n facilities and n locations are given, together with two $n \times n$ matrices $A = [a_{uv}]$ and $B = [b_{ij}]$, where a_{uv} is the *distance* between locations u and v, and b_{ij} is the *flow* between facilities i and j. A solution s is an assignment of each facility to a location. Let $x_i(s)$ denote the location to which facility i is assigned. The goal is to find an assignment that minimizes the function:

$$f(s) = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} a_{x_i(s) x_j(s)}.$$

1) Transformation of units: If all distance are multiplied by a constant ζ_1 and all flows by a constant ζ_2, the resulting instance $\bar{I}$ is equivalent to the original I, that is, $\bar{I} = g_1 I$, with $g_1 = \zeta_1 \zeta_2$.

2) Reference solution: The construction of the reference solution is typically stochastic: a number of solutions are randomly generated and improved through a local search. The best solution obtained is adopted as the reference solution [11]. It is worth noting that a local search is an invariant algorithm.

3) Heuristic information: Often, the heuristic information is not adopted [11], that is, $\beta = 0$. In this case, Condition 1 is trivially met. Some authors [12, 13] set $\eta_{ij} = 1/\sum_{l=1}^{n} a_{il}$. This meets Condition 1 with $g_2 = 1/\zeta_1$.

Therefore, the theorems on the weak invariance of ant system, $\mathcal{MAX}$–$\mathcal{MIN}$ ant system, and ant colony system hold.

4) Strongly-invariant heuristic information: If the heuristic information is adopted, $\eta_{ij} = f(s_0)/\sum_{l=1}^{n} a_{il}$, for all $\langle i, j \rangle$. This meets Condition 2 with $\lambda = f(s_0)$, and Condition 3. On the other hand, if no heuristic information is adopted as suggested in [11], Conditions 2 and 3 are trivially met.

Therefore, *si*AS, *si*$\mathcal{MM}$AS, and *si*ACS are indeed strongly invariant and are functionally equivalent to their original counterparts.

5 Generalized assignment problem

In the generalized assignment problem (GAP), a set of tasks I has to be assigned to a set of agents J in such a way that a cost function is minimized. Each agent j has only a limited capacity a_j, and each task i consumes, when assigned to agent j, a quantity b_{ij} of the agent's capacity. Moreover, a cost c_{ij} of assigning task i to agent j is given. The objective is to find a feasible task assignment s that minimizes

$$f(s) = \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij}(s), \text{ with } \sum_{i \in I} b_{ij} x_{ij}(s) \leq a_j, \forall j \in J.$$

$x_{ij}(s)$ is equal to 1 if task i is assigned to agent j in s, and 0 otherwise.

1) Transformation of units: If the cost of assigning each task i to each agent j is multiplied by a constant ζ, the resulting instance $\bar{I}$ is equivalent to the original I, that is, $\bar{I} = g_1 I$, with $g_1 = \zeta$.

2) Reference solution: The construction of the reference solution is typically stochastic.

3) Heuristic information: The typical setting is $\eta_{ij} = 1/c_{ij}$, for all $\langle i, j\rangle$ [14]. This meets Condition 1 with $g_2 = 1/\zeta$.

Therefore, the theorems on the weak invariance of ant system, $\mathcal{MAX}$–$\mathcal{MIN}$ ant system, and ant colony system hold.

4) Strongly-invariant heuristic information: Let $\eta_{ij} = f(s_0)/nc_{ij}$, for all $\langle i, j\rangle$, where $n = |N|$. This definition meets Condition 2 with $\lambda = f(s_0)/n$, and Condition 3.

Therefore, *si*AS, *si*$\mathcal{MM}$AS, and *si*ACS are indeed strongly invariant and are functionally equivalent to their original counterparts.

6 Single machine total weighted tardiness scheduling problem

In the single machine total weighted tardiness scheduling problem (SMTWTP) [15], n jobs are to be processed sequentially on a single machine, without interruption. Each job j has an associated processing time p_j, a weight w_j, and a due date d_j. All jobs are available for processing at time zero. The tardiness of job j is defined as $t_j(s) = \max\{0, c_j(s) - d_j\}$, where $c_j(s)$ is its completion time in the current job sequence s. The objective is to find the sequence s that minimizes the sum of the weighted tardiness:

$$f(s) = \sum_{j=1}^{n} w_j t_j(s).$$

1) Transformation of units: If, for all the jobs, both the *processing time* and the *due date* are multiplied by a constant ζ_1, and the weight is multiplied by a constant ζ_2, the resulting instance $\bar{I}$ is equivalent to the original I, that is, $\bar{I} = g_1 I$, with $g_1 = \zeta_1\zeta_2$.

2) Reference solution: A heuristic based on the *apparent urgency* [16] is typically used for the construction of the reference solution. In this approach, jobs are arranged in the descending order of their apparent urgency priorities:

$$AU_j = \frac{w_j}{p_j} \exp\left(\frac{-max\{0, d_j - t - p_j\}}{k\hat{p}}\right), \quad \text{for all } j.$$

Here, k is called the look-ahead parameter and is set according to the tightness of the due date; $\hat{p}$ the average processing time; t is the current time. As it can be seen, the selection criterion is invariant to transformation of units.

3) Heuristic information: The typical setting is $\eta_{ij} = 1/d_{ij}$, for all $\langle i, j\rangle$. This meets Condition 1 with $g_2 = 1/\zeta_1$.

Therefore, the theorems on the weak invariance of ant system, $\mathcal{MAX}$–$\mathcal{MIN}$ ant system, and ant colony system hold. In the literature, den Besten et al. [17] apply ant colony system to SMTWTP. In the analysis, the authors define various priority rules, which imply various heuristic measures. Among them the one previously described is considered.

4) Strongly-invariant heuristic information: Let $\eta_{ij} = \sum_{j=1}^{n} t_j(s_0)/d_{ij}$, for all $\langle i, j\rangle$. This definition meets Condition 2 with $\lambda = \sum_{j=1}^{n} t_j(s_0)$, and Condition 3.

Therefore, *si*AS, *si*$\mathcal{MM}$AS, and *si*ACS are indeed strongly invariant and are functionally equivalent to their original counterparts.

7 Open shop scheduling problem

In open shop scheduling problems (OSP) [18], a finite set $\mathcal{O}$ of operations is given, which is partitioned into a collection of subsets $\mathcal{M} = \{M_1, M_2, \ldots, M_U\}$ and a collection of subsets $\mathcal{J} = \{J_1, J_2, \ldots, J_V\}$. Each M_u is the set of operations that have to be performed by machine u; and each J_v is the set of operations belonging to job v. A non-negative *processing time* $t(o_j)$ and the *earliest possible starting time* $e(o_j)$ are associated with operation $o_j \in \mathcal{O}$. A solution s is a collection of schedules $\mathcal{X}(s) = \{X^1(s), X^2(s), \ldots, X^U(s)\}$, where $X^u(s)$ is the sequence of operations scheduled for machine u and $X^u_r(s)$ is the operation in position r in sequence $X^u(s)$. The completion time $c^u_r(s)$ of operation $X^u_r(s)$ is computed recursively from $c^u_{r'}(s) = t\big(X^u_{r'}(s)\big) + \max\big[e\big(X^u_{r'}(s)\big), c^u_{r'-1}(s)\big]$, with $c^u_0(s) = 0$. The goal is to minimize the *makespan*, which is given by:

$$f(s) = \max_u c^u_{|M_u|}(s).$$

1) Transformation of units: If all processing times and earliest possible starting times are multiplied by a constant ζ, the resulting instance $\bar{I}$ is equivalent to the original I, that is, $\bar{I} = g_1 I$, with $g_1 = \zeta$.

2) Reference solution: The construction of the reference solution is typically stochastic.

3) Heuristic information: The heuristic information is $\eta_{ij} = 1/e(o_j)$, for all $\langle i, j\rangle$, which meets Condition 1 with $g_2 = 1/\zeta$.

Therefore, the theorems on the weak invariance of ant system, $\mathcal{MAX}$–$\mathcal{MIN}$ ant system, and ant colony system hold.

4) Strongly-invariant heuristic information: The heuristic information is $\eta_{ij} = f(s_o)/e(o_j)$, for all $\langle i, j\rangle$. This meets Condition 2 with $\lambda = f(s_o)$, and Condition 3.

Therefore, *si*AS, *si*$\mathcal{MM}$AS, and *si*ACS are indeed strongly invariant and are functionally equivalent to their original counterparts.

8 Permutation flow shop scheduling problem

In the Permutation flow shop scheduling problem (PFSP), J jobs are to be processed. A set I of machines is available. Each job j is partitioned in a set of operations $O_j = \{o_{1j}, o_{2j}, \cdots, o_{Ij}\}$, where operation o_{ij} is to be processed on machine i. To each operation o_{ij} a *processing time* $t(i, j)$ is associated. It may be equal to 0 if job j is not to be processed on machine i. Let $x_r(s)$ be the r-th job scheduled in sequence s. The completion time $c^u_r(s)$ of operation $o_{ux_r(s)}$ on machine u is computed recursively from $c^u_{r'}(s) = t\big(u, o_{ux_{r'}(s)}\big) + c^u_{r'-1}(s)$, with $c^u_0(s) = 0$.

The goal is to minimize the *makespan*:

$$f(s) = \max_u c^u_{|J|}(s).$$

1) Transformation of units: If the processing time is multiplied by a constant ζ for all the jobs, the resulting instance $\bar{I}$ is equivalent to the original I, that is, $\bar{I} = g_1 I$, with $g_1 = \zeta$.

2) Reference solution: The NEH heuristic [16] is typically used for the construction of the reference solution. In this approach, first the jobs are ordered by decreasing sums of total job processing times on the machines. Then, the first two jobs are scheduled so as to minimize the partial makespan as if there were only two jobs. Finally, for all the following ones, insert one job at a time in the partial schedule, into the location which minimizes the partial makespan. As it can be seen, the selection criterion is invariant to transformation of units.

3) Heuristic information: Typically, the heuristic information is not adopted [19], that is, $\beta = 0$. In this case, Condition 1 is trivially met.

Therefore, the theorems on the weak invariance of ant system, $\mathcal{MAX}$–$\mathcal{MIN}$ ant system, and ant colony system hold.

4) Strongly-invariant heuristic information: As for the case seen in 3, if no heuristic information is adopted as suggested by Stützle [19], Conditions 2 and 3 are trivially met.

Therefore, *si*AS, *si*$\mathcal{MM}$AS, and *si*ACS are indeed strongly invariant and are functionally equivalent to their original counterparts.

9 Set covering problem

In the set covering problem (SCP) [20], $u \times v$ matrix $A = [a_{ij}]$ is given. All the matrix element are either 0 or 1. Additionally, each column is given a non-negative cost c_j. We say that a column j covers a row i if $a_{ij} = 1$. A solution s is represented by a subset of columns that covers every row. Let $x_j(s)$ be a binary variable which is 1 if column j is included in s, and 0 otherwise. The goal is finding a solution of minimal cost. The objective function is:

$$f(s) = \sum_{j=1}^{v} c_j x_j(s).$$

1) Transformation of units: If the cost of all columns is multiplied by a constant ζ, the resulting instance $\bar{I}$ is equivalent to the original I, that is, $\bar{I} = g_1 I$, with $g_1 = \zeta$.

2) Reference solution: The construction of the reference solution is typically stochastic.

3) Heuristic information: Let $\eta_j = e_j/c_j$, where e_j is the *cover value* of column j, that is, the number of additional rows covered when adding column j to the current partial solution. This meets Condition 1 with $g_2 = 1/\zeta$.

Therefore, the theorems on the weak invariance of ant system, $\mathcal{MAX}$–$\mathcal{MIN}$ ant system, and ant colony system hold. Ant system is applied to the set covering problem by Leguizamón and Michalewicz [21], Hadji et al. [22], using the invariant framework.

4) Strongly-invariant heuristic information: Let $\eta_j = f(s_o)e_j/c_j$. This meets Condition 2 with $\lambda = f(s_o)$, and Condition 3.

Therefore, *si*AS, *si*$\mathcal{MM}$AS, and *si*ACS are indeed strongly invariant and are functionally equivalent to their original counterparts.

10 Arc-weighted k-cardinality tree problem

The arc-weighted k-cardinality tree problem (KCT) is a generalization of the minimum spanning tree problem. It consists in finding a subtree with exactly k arcs in a graph with arc weights, such that the sum of the weights is minimal [23].

More formally, the KCT problem can be defined as follows. Let $G = (N, E)$ be a graph. A weight c_{ij} is assigned to each edge $\langle i, j\rangle$. A solution s is a k-cardinality tree in G. Let $x_{ij}(s)$ be a binary variable which is 1 if edge $\langle i,j\rangle$ is included in s, and 0 otherwise. Then, the edge-weighted problem (G,c,k) consists of finding a solution s that minimizes the objective function:

$$f(s) = \sum_{\langle i,j\rangle \in E} c_{ij}x_{ij}(s).$$

1) Transformation of units: If the weight of all edges is multiplied by a constant ζ, the resulting instance $\bar{I}$ is equivalent to the original I, that is, $\bar{I} = g_1 I$, with $g_1 = \zeta$. Indeed, $\bar{c}_{ij} = \zeta c_{ij}$, for all $j \implies \bar{f}(s) = \zeta f(s)$, for all s.

2) Reference solution: The construction of the reference solution is typically stochastic.

3) Heuristic information: The heuristic information typically used [24] is $\eta_{ij} = 1/c_{ij}$. This meets Condition 1 with $g_2 = 1/\zeta$.

Therefore, the theorems on the weak invariance of ant system, $\mathcal{MAX}$-$\mathcal{MIN}$ ant system, and ant colony system hold.

4) Strongly-invariant heuristic information: Let $\eta_{ij} = f(s_0)/c_{ij}$. This meets Condition 2 with $\lambda = f(s_0)$, and Condition 3.

Therefore, *si*AS, *si*$\mathcal{MM}$AS, and *si*ACS are indeed strongly invariant and are functionally equivalent to their original counterparts.

11 Multiple knapsack problem

In the multiple knapsack problem (MKP), a set I of items and a set J of resources are given. A profit p_i and a requirement r_{ij} of resource $j \in J$ are assigned to each item $i \in I$. A set of constraints is given as a limit a_j on each resource j [25]. A solution s is a subset of items that meets all the constraints. The goal is to find a solution that maximizes the total profit. The objective function can be formulate as:

$$f(s) = \sum_{i\in I} p_i x_i(s), \text{ with } \sum_{i\in I} r_{ij}x_i(s) \le a_j, \forall j \in J.$$

$x_i(s)$ is a binary variable which is 1 if $i \in s$, and 0 otherwise.

1) Transformation of units: If the profit associated to each item is multiplied by a constant ζ, the resulting instance $\bar{I}$ is equivalent to the original I, that is, $\bar{I} = g_1 I$, with $g_1 = \zeta$.

2) Reference solution: The construction of the reference solution is typically stochastic.

3) Heuristic information: Let $s_k(t)$ be the partial solution of ant k at construction step t, and $v_j(k,t) = a_j - \sum_{z\in s_k(t)} r_{zj}$ be the remaining amount of resource j. The tightness of component i with respect to resource j is defined as: $w_{ij}(k,t) = r_{ij}/v_j(k,t)$. Finally the average tightness of all constraints with respect to component i is computed as $aw_i(k,t) = \sum_{j\in R} w_{ij}(k,t)/l$, where l is the number of resource constraints. Let $\eta_i(s_k(t)) = p_i/aw_i(k,t)$ [26]. This definition meets Condition 1 with $g_2 = \zeta$.

Therefore, the theorems on the weak invariance of ant system, $\mathcal{MAX}$-$\mathcal{MIN}$ ant system, and ant colony system hold.

4) Strongly-invariant heuristic information: Let $\eta_i(s_k(t)) = p_i/\left(aw_i(k,t)f(s_0)\right)$. This meets Condition 2 with $\lambda = 1/f(s_0)$, and Condition 3.

Therefore, *si*AS, *si*$\mathcal{MM}$AS, and *si*ACS are indeed strongly invariant and are functionally equivalent to their original counterparts.

12 Bin packing problem

In the bin packing problem (BPP), a set J of bins of fixed capacity W and a set I of items are given. A fixed weight w_i, $0 < w_i < W$ is associated to each item i [27]. A solution s is an assignment of the items to bins such that the capacity of the bins is not violated. The goal is to pack the items in as few bins as possible. More formally, let $x_{ij}(s)$ be a binary variable which is 1 if item i is inserted in bin j in solution s, and 0 otherwise. Let $y_j(s)$ be a binary variable which is 1 if bin j is used is solution s, that is, if $\sum_{i \in I} x_{ij}(s) > 0$; and 0 otherwise. The function to be minimized is:

$$f(s) = \sum_{j \in J} y_j(s), \text{ with } \sum_{i \in I} w_i x_{ij}(s) < W, \forall j \in J.$$

1) Transformation of units: If the capacity of the bins and the weight of all the items are multiplied by a constant ζ, the resulting instance $\bar{I}$ is equivalent to the original I, that is, $\bar{I} = g_1 I$, with $g_1 = 1$.

2) Reference solution: The construction of the reference solution is typically done using the best fit decreasing heuristic. In this approach, first the items are ordered by decreasing weight. Then, each of them is assigned to the bin which will have the least amount of space left, after accommodating the item. As it can be seen, the selection criterion is invariant to transformation of units.

3) Heuristic information: The heuristic information is typically defined as $\eta_{ij} = 1/w_i$ [28]. This meets Condition 1 with $g_2 = \zeta$.

Therefore, the theorems on the weak invariance of ant system, $\mathcal{MAX}$–$\mathcal{MIN}$ ant system, and ant colony system hold.

4) Strongly-invariant heuristic information: Let $\eta_{ij} = f(s_0)/w_i$. This meets Condition 2 with $\lambda = 1/f(s_0)$, and Condition 3.

Therefore, *si*AS, *si*$\mathcal{MM}$AS, and *si*ACS are indeed strongly invariant and are functionally equivalent to their original counterparts.

Conclusions

In a recent study, Birattari et al. [1] have shown that three main ant colony optimization algorithms – ant system, ant colony system and $\mathcal{MAX}$–$\mathcal{MIN}$ ant system– are invariant to transformations of the cost unit used for expressing the instances. For this characteristic to be realized, a condition concerning the heuristic information needs to be met. Moreover, the three algorithms are shown to be implementable in a strongly invariant fashion in case two more conditions are satisfied.

In this study we presented a short list of problems to which ant colony optimization algorithms can be applied satisfying the three conditions. From this short list, it can be observed that ACO can enjoy this advantage with many typical combinatorial optimization problems.

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