Université Libre de Bruxelles

Institut de Recherches Interdisciplinaires et de Développements en Intelligence Artificielle

On the Invariance of Ant System

Mauro Birattari, Paola Pellegrini, and Marco Dorigo

IRIDIA – Technical Report Series

Technical Report No.
TR/IRIDIA/2005-036

December 2005

IRIDIA – Technical Report Series
ISSN 1781-3794

Published by:
IRIDIA, *Institut de Recherches Interdisciplinaires et de Développements en Intelligence Artificielle*
Université Libre de Bruxelles
Av F. D. Roosevelt 50, CP 194/6
1050 Bruxelles, Belgium

Technical report number TR/IRIDIA/2005-036

On the Invariance of Ant System

Mauro Birattari, Paola Pellegrini, and Marco Dorigo

December 2005

The *hyper-cube framework* [1] has been recently introduced with the aim of implementing ant colony optimization algorithms [2] that are invariant with respect to a linear rescaling of problem instances. The need for the introduction of the hyper-cube framework has been explicitly motivated by the observation that

> *in standard ACO algorithms the pheromone values and therefore the performance of the algorithms, strongly depend on the scale of the problem.* [1]

In this short paper, we formally show that this statement is only partially correct: Indeed, in standard ant colony optimization algorithms the pheromone trail (and the heuristic values) depend on the scale of the problem. Nonetheless, the sequence of solutions they find is independent from the scaling.

For definiteness, the paper focuses on ant system for the traveling salesman problem. The theorems we enunciate in the paper are proved for this specific algorithm and for this specific problem. The conditions under which these results extend to other problems are discussed in the following.

Moreover, the paper proposes a trivial modification of ant system that, beside being able to find the same solutions irrespectively of the scaling, has also the property that the pheromone and the heuristic values do not depend on the scaling itself.

Definition 1 (Random proportional rule). At the generic iteration h, given that ant k is in node i and that the set of feasible nodes is $\mathcal{N}_i^k$, the node j to which ant k moves is selected with probability:

$$p_{ij,h}^k = \frac{[\tau_{ij,h}]^\alpha [\eta_{ij}]^\beta}{\sum_{l \in \mathcal{N}_i^k} [\tau_{il,h}]^\alpha [\eta_{il}]^\beta}, \tag{1}$$

where α and β are parameters, $\tau_{ij,h}$ is the pheromone value associated with arc $\langle i, j\rangle$ at iteration h, and η_{ij} is a piece of *heuristic information* on the desirability of visiting node j after node i.

Definition 2 (Heuristic information). When solving the traveling salesman problem, the heuristic information η_{ij} is the inverse of the cost of traveling from city i to city j:

$$\eta_{ij} = \frac{1}{c_{ij}}. \tag{2}$$

Definition 3 (Pheromone update rule). At the generic iteration h, given that m ants have generated the solutions $T_h^1, T_h^2, \dots, T_h^m$ of cost $C_h^1, C_h^2, \dots, C_h^m$, respectively, the pheromone is updated according to the following rule:

$$\tau_{ij,h+1} = (1-\rho)\tau_{ij,h} + \sum_{k=1}^{m} \Delta_{ij,h}^k, \tag{3}$$

where ρ is a parameter called *evaporation rate* and

$$\Delta_{ij,h}^k = \begin{cases} 1/C_h^k, & \text{if } \langle i, j\rangle \in T_h^k; \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Definition 4 (Ant system). Ant system is an ant colony optimization algorithm in which the solutions are constructed according to the random proportional rule given in Definition 1, and the pheromone is updated according to the rule given in Definition 3. The evaporation rate ρ, the number of ants m, and the exponents α and β are parameters of the algorithm.

When ant system is used for solving the traveling salesman problem, it is customary to initialize the pheromone as follows.

Definition 5 (Nearest-neighbor pheromone initialization). The pheromone on all arcs $\langle i, j\rangle$ is initialized to the same value:

$$\tau_{ij,0} = \frac{m}{C^{nn}}, \tag{5}$$

where m is the number of ants generated at each iteration and C^{nn} is the cost of the solution T^{nn} obtained by the nearest-neighbor heuristic.

The following theorems hold true.

Lemma 1. *The random proportional rule is invariant to linear transformations of the pheromone. Formally:*

$$\bar{\tau}_{ij,h} = g\tau_{ij,h} \implies \bar{p}^k_{ij,h} = p^k_{ij,h}, \tag{6}$$

where $\bar{p}^k_{ij,h}$ is obtained on the basis of $\bar{\tau}_{ij,h}$, according to Definition 1.

Proof. According to Definition 1:

$$\begin{aligned}\bar{p}^k_{ij,h} &= \frac{[\bar{\tau}_{ij,h}]^\alpha[\eta_{ij}]^\beta}{\sum_{l\in\mathcal{N}_i^k}[\bar{\tau}_{il,h}]^\alpha[\eta_{il}]^\beta} = \frac{[g\tau_{ij,h}]^\alpha[\eta_{ij}]^\beta}{\sum_{l\in\mathcal{N}_i^k}[g\tau_{il,h}]^\alpha[\eta_{il}]^\beta} = \frac{[g]^\alpha[\tau_{ij,h}]^\alpha[\eta_{ij}]^\beta}{\sum_{l\in\mathcal{N}_i^k}[g]^\alpha[\tau_{il,h}]^\alpha[\eta_{il}]^\beta} \\ &= \frac{[g]^\alpha[\tau_{ij,h}]^\alpha[\eta_{ij}]^\beta}{[g]^\alpha\sum_{l\in\mathcal{N}_i^k}[\tau_{il,h}]^\alpha[\eta_{il}]^\beta} = \frac{[\tau_{ij,h}]^\alpha[\eta_{ij}]^\beta}{\sum_{l\in\mathcal{N}_i^k}[\tau_{il,h}]^\alpha[\eta_{il}]^\beta} = p^k_{ij,h}.\end{aligned} \tag{7}$$

□

Lemma 2. *The random proportional rule is invariant to linear transformations of the heuristic information. Formally:*

$$\bar{\eta}_{ij} = g\eta_{ij} \implies \bar{p}^k_{ij,h} = p^k_{ij,h}, \tag{8}$$

where $\bar{p}^k_{ij,h}$ is obtained on the basis of $\bar{\eta}_{ij}$, according to Definition 1.

Proof. The proof follows closely the one given for Lemma 1:

$$\begin{aligned}\bar{p}^k_{ij,h} &= \frac{[\tau_{ij,h}]^\alpha[\bar{\eta}_{ij}]^\beta}{\sum_{l\in\mathcal{N}_i^k}[\tau_{il,h}]^\alpha[\bar{\eta}_{il}]^\beta} = \frac{[\tau_{ij,h}]^\alpha[g\eta_{ij}]^\beta}{\sum_{l\in\mathcal{N}_i^k}[\tau_{il,h}]^\alpha[g\eta_{il}]^\beta} = \frac{[g]^\beta[\tau_{ij,h}]^\alpha[\eta_{ij}]^\beta}{\sum_{l\in\mathcal{N}_i^k}[g]^\beta[\tau_{il,h}]^\alpha[\eta_{il}]^\beta} \\ &= \frac{[g]^\beta[\tau_{ij,h}]^\alpha[\eta_{ij}]^\beta}{[g]^\beta\sum_{l\in\mathcal{N}_i^k}[\tau_{il,h}]^\alpha[\eta_{il}]^\beta} = \frac{[\tau_{ij,h}]^\alpha[\eta_{ij}]^\beta}{\sum_{l\in\mathcal{N}_i^k}[\tau_{il,h}]^\alpha[\eta_{il}]^\beta} = p^k_{ij,h}.\end{aligned} \tag{9}$$

□

Lemma 3. *The random proportional rule is invariant to concurrent linear transformations of the pheromone and of the heuristic information. Formally:*

$$\bar{\tau}_{ij,h} = g_1\tau_{ij,h} \wedge \bar{\eta}_{ij} = g_2\eta_{ij} \implies \bar{p}^k_{ij,h} = p^k_{ij,h}. \tag{10}$$

where $\bar{p}^k_{ij,h}$ is obtained on the basis of $\bar{\tau}_{ij,h}$ and $\bar{\eta}_{ij}$, according to Definition 1.

Proof. Similarly to the proofs of Lemmas 1 and 2:

$$\begin{aligned}\bar{p}^k_{ij,h} &= \frac{[\bar{\tau}_{ij,h}]^\alpha[\bar{\eta}_{ij}]^\beta}{\sum_{l\in\mathcal{N}_i^k}[\bar{\tau}_{il,h}]^\alpha[\bar{\eta}_{il}]^\beta} = \frac{[g_1\tau_{ij,h}]^\alpha[g_2\eta_{ij}]^\beta}{\sum_{l\in\mathcal{N}_i^k}[g_1\tau_{il,h}]^\alpha[g_2\eta_{il}]^\beta} = \frac{[g_1]^\alpha[g_2]^\beta[\tau_{ij,h}]^\alpha[\eta_{ij}]^\beta}{\sum_{l\in\mathcal{N}_i^k}[g_1]^\alpha[g_2]^\beta[\tau_{il,h}]^\alpha[\eta_{il}]^\beta} \\ &= \frac{[g_1]^\alpha[g_2]^\beta[\tau_{ij,h}]^\alpha[\eta_{ij}]^\beta}{[g_1]^\alpha[g_2]^\beta\sum_{l\in\mathcal{N}_i^k}[\tau_{il,h}]^\alpha[\eta_{il}]^\beta} = \frac{[\tau_{ij,h}]^\alpha[\eta_{ij}]^\beta}{\sum_{l\in\mathcal{N}_i^k}[\tau_{il,h}]^\alpha[\eta_{il}]^\beta} = p^k_{ij,h}.\end{aligned} \tag{11}$$

□

Definition 6 (Linear transformation of a TSP instance). With $\bar{I} = fI$ we indicate that the instance $\bar{I}$ is a linear transformation of the instance I: The two instances have the same number of cities and the cost $\bar{c}_{ij}$ of traveling from city i to city j in $\bar{I}$ is f times the corresponding cost in instance I. Formally:

$$\bar{c}_{ij} = fc_{ij}, \quad \text{for all arcs } \langle i, j\rangle. \tag{12}$$

Remark 1. The cost $\bar{C}$ of a solution $\bar{T}$ of instance $\bar{I}$ is f times the cost C of the corresponding solution T of instance I. Formally:

$$\bar{I} = fI \wedge \bar{T} = T \implies \bar{C} = fC. \tag{13}$$

Definition 7 (Invariance). An ant colony optimization algorithm is **invariant** to linear transformations if the sequence of solutions S_I generated when solving an instance I and the sequence of solutions $S_{\bar{I}}$ generated when solving an instance $\bar{I}$ are the same whenever $\bar{I}$ is a linear transformation of I.

Definition 8 (Strong and weak invariance). An ant colony optimization algorithm is said to be **strongly-invariant** if, beside generating the same solutions on any two linearly related instances I and $\bar{I}$, it also enjoys the property that the heuristic information and the pheromone at each iteration are the same when solving I and $\bar{I}$. Conversely, the algorithm is **weakly-invariant** if it obtains the same solutions on linearly related instances but the heuristic information and the pheromone assume different values.

Hypothesis 1 (Pseudo-random number generator). When solving two instances I and $\bar{I}$, the stochastic decisions taken while constructing solutions are made on the basis of random experiments based on pseudo-random numbers produced by the same random number generator. We assume that this generator is initialized with the same seed when solving the two instances so that the two sequences of random numbers that are generated are the same in the two cases.

Theorem 1. *Ant system is weakly-invariant.*[1]

Proof. Let us consider two generic instances I and $\bar{I}$ such that

$$\bar{I} = fI. \tag{14}$$

According to Definition 2, and taking into account Equation 12, it results:

$$\bar{\eta}_{ij} = \frac{1}{f}\eta_{ij}. \tag{15}$$

According to Lemma 3, if $\bar{\tau}_{ij,h} = \frac{1}{f}\tau_{ij,h}$, at the generic iteration h, then $\bar{p}^k_{ij,h} = p^k_{ij,h}$. Under Hypothesis 1,

$$\bar{T}^k_h = T^k_h, \quad \text{for all } k = 1, \ldots, m, \tag{16}$$

and therefore, according to Equation 13,

$$\bar{C}^k_h = fC^k_h, \quad \text{for all } k = 1, \ldots, m. \tag{17}$$

[1]We insist that, for definiteness, we refer here to the application of ant system to the traveling salesman problem and we consider the case in which the pheromone is initialized as prescribed by Definition 5.

According to Equation 4:

$$
\begin{aligned}
\bar{\Delta}^k_{ij,h} &= \begin{cases} 1/\bar{C}^k_h, & \text{if } \langle i,j\rangle \in \bar{T}^k_h; \\ 0, & \text{otherwise} \end{cases} = \begin{cases} 1/fC^k_h, & \text{if } \langle i,j\rangle \in \bar{T}^k_h = T^k_h; \\ 0/f, & \text{otherwise} \end{cases} \\
&= \frac{1}{f}\begin{cases} 1/C^k_h, & \text{if } \langle i,j\rangle \in T^k_h; \\ 0, & \text{otherwise} \end{cases} = \frac{1}{f}\Delta^k_{ij,h},
\end{aligned}
\tag{18}
$$

and therefore:

$$
\begin{aligned}
\bar{\tau}_{ij,h+1} &= (1-\rho)\bar{\tau}_{ij,h} + \sum_{k=1}^{m}\bar{\Delta}^k_{ij,h} = (1-\rho)\frac{1}{f}\tau_{ij,h} + \sum_{k=1}^{m}\frac{1}{f}\Delta^k_{ij,h} \\
&= (1-\rho)\frac{1}{f}\tau_{ij,h} + \frac{1}{f}\sum_{k=1}^{m}\Delta^k_{ij,h} = \frac{1}{f}\left((1-\rho)\tau_{ij,h} + \sum_{k=1}^{m}\Delta^k_{ij,h}\right) = \frac{1}{f}\tau_{ij,h+1}.
\end{aligned}
\tag{19}
$$

In order to provide a basis for the above defined recursion and therefore to conclude the proof, it is sufficient to observe that at $h=0$ the pheromone is initialized as:

$$
\bar{\tau}_{ij,0} = \bar{\tau}_0 = \frac{m}{\bar{C}^{nn}} = \frac{m}{fC^{nn}} = \frac{1}{f}\tau_{ij,0}. \tag{20}
$$

□

Remark 2. Theorem 1 holds true for any way of initializing the pheromone, provided that for any two instances $\bar{I}$ and I such that $\bar{I} = fI$, $\bar{\tau}_{ij,0} = \frac{1}{f}\tau_{ij,0}$, for all arcs $\langle i,j\rangle$.

Remark 3. Theorem 1 extends to the application of ant system to problems other than the traveling salesman problem, provided that the initialization of the pheromone is performed as prescribed in Remark 2 and for any two instances $\bar{I}$ and I such that $\bar{I} = fI$, there exists a coefficient $g \neq 0$ such that $\bar{\eta}_{ij} = g\eta_{ij}$, for all arcs $\langle i,j\rangle$. In particular, it is worth noting here that one notable case in which this last condition is satisfied is when $\bar{\eta}_{ij} = \eta_{ij} = 0$, for all arcs $\langle i,j\rangle$, that is, when no heuristic information is used.

Strongly-Invariant Ant System

A strongly invariant version of ant system (siAS) can be easily defined. For definiteness, we present here a version of siAS for the traveling salesman problem.

Definition 9 (Strongly-invariant heuristic information). When solving the traveling salesman problem, the heuristic information η_{ij} is

$$
\eta_{ij} = \frac{C^{nn}}{c_{ij}}, \tag{21}
$$

where c_{ij} is the cost of traveling from city i to city j and C^{nn} is the cost of the solution T^{nn} obtained by the nearest-neighbor heuristic.

Definition 10 (Strongly-invariant pheromone update rule). The pheromone is updated using the same rule given in Definition 3, with the only difference that $\Delta^k_{ij,h}$ is given by:

$$
\Delta^k_{ij,h} = \begin{cases} C^{nn}/C^k_h, & \text{if } \langle i,j\rangle \in T^k_h; \\ 0, & \text{otherwise;} \end{cases} \tag{22}
$$

where C^{nn} is the cost of the solution T^{nn} obtained by the nearest-neighbor heuristic.

Definition 11 (Strongly-invariant pheromone initialization). The pheromone on all arcs $\langle i,j\rangle$ is initialized to the same value:

$$
\tau_{ij,0} = m, \tag{23}
$$

where m is the number of ants generated at each iteration.

Definition 12 (Strongly-invariant ant system). The strongly-invariant ant system (siAS) is a minor variation of ant system. It shares with ant system the random proportional rule for the construction of solutions, but in siAS the heuristic values are set as in Definition 9, the pheromone is initialized according to Definition 11 and the update is performed according to Definition 10.

Remark 4. In the definition of siAS given above, the nearest-neighbor heuristic has been adopted for generating a reference solution, the cost of which is then used for normalizing the cost of the solutions found by the algorithm. Any other mechanism could be used instead of the nearest-neighbor heuristic, provided that the solution this algorithm returns does not depend on the scale of the problem.

Acknowledgments. This work was supported by the ANTS project, an *Action de Recherche Concertée* funded by the Scientific Research Directorate of the French Community of Belgium. Marco Dorigo acknowledges support from the Belgian National Fund for Scientific Research (FNRS), of which he is a Research Director.

References

[1] C. Blum and M. Dorigo. The hyper-cube framework for ant colony optimization. *IEEE Transactions on Systems, Man, and Cybernetics—Part B*, 34(2):1161–1172, 2004.

[2] M. Dorigo and T. Stützle. *Ant Colony Optimization.* MIT Press, Cambridge, MA, USA, 2004.