On Conditional Belief Function Independence

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Extended Abstract.

The concept of conditional independence has been extensively studied in probability theory (see, for instance, [2], [3], [6], ...). Pearl and Paz [7] have introduced some basic properties of the conditional independence relation, called "*graphoid axioms*". These axioms are satisfied not only by probabilistic conditional independence, but also by embedded multi-valued dependency models in relational databases [8], by conditional independence in Spohn's theory of ordinal conditional functions [11], [4], by qualitative conditional independence in Dempster-Shafer theory of belief functions partitions [9], and by conditional independence in valuation-based systems (VBS) [10] capable of representing many different uncertainty calculi.

The aim of this paper is to propose the new definitions of conditional independence when uncertainty is expressed under the form of belief functions and then to discuss the relationships between these definitions. The notion of conditional independence is given with the *conditional independence relations* [6], [2], [3], which successfully depict our intuition about how dependencies should update in response to new pieces of information.

This paper is organized as follows: we first recall the definition of probabilistic conditional independence. Then, after extending the definition of evidential and cognitive independence to the conditional case, we present our definitions of conditional non-interactivity (section 1.3), conditional irrelevance (section 1.4) and conditional doxastic independence (section 1.5) for belief functions. Finally, we present an axiomatic characterization for conditional belief functions independence relations.

1 Conditional Belief Function Independence Concepts

The definitions of marginal independence for belief functions presented in Ben Yaghlane *et al.* [1] can be extended to the case of conditional independence.

1.1 Probabilistic Conditional Independence

First, we present a meaning of the conditional independence concept in the probability theory. For random variables X, Y, Z, and P a distribution on the space $\Theta_X \times \Theta_Y \times \Theta_Z$ (or simply XYZ), we may write $X \amalg_P Y \mid Z$ to denote that, X and Y are *conditionally independent* given Z, with respect to P.

The usual definition of $X \amalg_P Y \mid Z$ is in terms of the factorization of the conditional joint probability distribution of (X, Y) given Z.

There is another equivalent definition, which is more intuitive. This definition can be interpreted as *conditional irrelevance* and it means that once the value of Z is specified, any further information about Y is irrelevant to the uncertainty about X.

1.2 Evidential and Cognitive Conditional Independence

Following Kong [5], we define the notion of strong conditional independence of belief functions as follows :

Definition 1 *Given three variables X, Y, and Z. We say that X and Y are conditionally independent given Z if and only if for all* $x \subseteq X$*, all* $y \subseteq Y, z \in Z$*:*

$$pl^{XYZ}(x, y) = pl^{XYZ\downarrow XZ}(x)pl^{XYZ\downarrow YZ}(y) \tag{1}$$
$$bel^{XYZ}(x, y) = bel^{XYZ\downarrow XZ}(x)bel^{XYZ\downarrow YZ}(y) \tag{2}$$

where $\mathrm{pl}^{XYZ\downarrow XZ}$ (resp. $\mathrm{bel}^{XYZ\downarrow XZ}$) and $\mathrm{pl}^{XYZ\downarrow YZ}$ (resp. $\mathrm{bel}^{XYZ\downarrow YZ}$) are carried by XZ and YZ, respectively.

The *weak* ("*cognitive*") *conditional independence* is derived straightforwardly if we only consider the equation (1).

1.3 Conditional Non-Interactivity

For the definition of conditional non-interactivity for belief functions, we start from computing the belief of joint product XYZ. We marginalize on XZ and also on YZ. We combine these two marginals XZ and YZ, and we want it to be equal to the initial one (on XYZ) combined with the marginal on Z. The formal definition is given as follows:

Definition 2 *Given three variables X, Y and Z, and m on XYZ. X and Y are non-interactive given Z with respect to m, denoted by* $X \perp_m Y \mid Z$*, if and only if*

$$m^{XYZ} \oplus m^{XYZ\downarrow Z} = m^{XYZ\downarrow XZ} \oplus m^{XYZ\downarrow YZ} \tag{3}$$

The equation (3) corresponds to Shenoy' factorization (see [10], lemma 3.1 (5) page 215). Note that Shenoy' definition considers that the terms $\mathrm{m}^{XYZ\downarrow XZ}$ and $\mathrm{m}^{XYZ\downarrow YZ}$ are arbitrary and not necessarily the marginals of XYZ on XZ and YZ, respectively. As a consequence, with Shenoy' definition, we loose the connection with the "common sense meaningful".

In addition, Studeny [12] notice that the definition of conditional belief function non-interactivity [1] is *not consistent with marginalization*. This means that it may happen for two bba's m_1 and m_2 on XZ and YZ, respectively, which are consonant (i.e. $\mathrm{m}_1^Z = \mathrm{m}_2^Z$) there exists no bba m on XYZ such that $\mathrm{m}^{XYZ\downarrow XZ} = \mathrm{m}_1$, $\mathrm{m}^{XYZ\downarrow YZ} = \mathrm{m}_2$ and $X \perp_m Y \mid Z$. We explain this objection by an example.

[1] Studeny uses the term "conditional independence" rather than "conditional non-interactivity"

1.4 Conditional Irrelevance

In order to present the definition of conditional irrelevance for belief functions, we first introduce a concept of *the set of m^{YZ} indistinguishable on Z under m^{XYZ}*.

Building the set

Given m on XYZ and any m^{YZ} on YZ
Let $m^* = (m \oplus m^{YZ})^{\downarrow Z}$
Find all m' on YZ so that $(m \oplus m')^{\downarrow Z} = m^*$
Denote this set as $R^Z(m^{YZ})$.

This family corresponds to the set of pairs of belief functions indistinguishable on Z under m^{XYZ}. The formal definition of this set is:

Definition 3 *For any m^{XYZ}, and any bba m_1, m_2, we have $(m_1, m_2) \in R^Z(m^{XYZ})$ iff $(m^{XYZ} \oplus m_1)^{\downarrow Z} = (m^{XYZ} \oplus m_2)^{\downarrow Z}$*

Once the set $R^Z(m^{XYZ})$ is built, we define the notion of conditional irrelevance as follows :

Definition 4 *Given three variables X, Y and Z, and m on XYZ. Y is irrelevant to X given Z with respect to m, denoted by $IR_m(X, Y \mid Z)$, if and only if $\forall m^{YZ}, \forall m' \in R^Z(m^{XYZ})$*

$$(m \oplus m')^{\downarrow XZ} \propto (m \oplus m^{YZ})^{\downarrow XZ} \tag{4}$$

1.5 Conditional Doxastic Independence

The notion of doxastic independence in conditional case can be defined as follows :

Definition 5 *Given three variables X, Y and Z, and m on XYZ. X and Y are doxastically independent given Z with respect to m, denoted by $X \amalg_m Y \mid Z$, if and only if m satisfies*

- $IR_m(X, Y \mid Z)$
- $\forall m_0$ *on XYZ* : $IR_{m_0}(X, Y \mid Z) \Rightarrow IR_{m \oplus m_0}(X, Y \mid Z)$

Theorem 1 *Given three variables X, Y and Z, and m on XYZ. X and Y are doxastically independent given Z with respect to m ($X \amalg_m Y \mid Z$) if and only if X and Y are non-interactive given Z with respect to m ($X \perp_m Y \mid Z$).*

2 Conditional Belief Function Independence Relations

The intuitive meaning of conditional independence is when we say that a random variable X is *independent* to Y given Z, denoted by $X \amalg Y \mid Z$, we mean that once the value of Z has been specified, any further information about Y is irrelevant to uncertainty about X.

The properties of conditional independence can be considered as a rules' set useful to infer new independence relations from an initial set. They are also important when we need a *graphical representation* of dependencies [6].

In this section, we present the conditional independence properties for belief functions. For the proofs, we use the definition of conditional non-interactivity.

Definition 6 *Let X, Y, Z and W be disjoint subsets of U, and a mass m over the product space. Then we define the following properties:*

$$\begin{array}{rl} Symmetry & X \perp_m Y \mid Z \Leftrightarrow Y \perp_m X \mid Z \\ Decomposition & X \perp_m Y \cup W \mid Z \Rightarrow X \perp_m Y \mid Z \\ WeakUnion & X \perp_m Y \cup W \mid Z \Rightarrow X \perp_m Y \mid W \cup Z \\ Contraction & X \perp_m Y \mid Z \text{ and } X \perp_m W \mid Y \cup Z \Rightarrow X \perp_m Y \cup W \mid Z \\ Intersection & X \perp_m Y \mid Z \text{ and } X \perp_m Z \mid Y \Rightarrow X \perp_m Y \cup Z \end{array}$$

The conditional non-interactivity relation satisfies symmetry, decomposition, weak union, contraction, and intersection. So it is a *graphoid*.

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