THE TRANSFERABLE BELIEF MODEL AND POSSIBILITY THEORY.

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4-page Abstract.

1) Two theories.

We want to show the difference between the degrees of possibilities (or necessities) and the degrees of belief (or plausibility). The overall principle is that the degrees of possibilities and necessities are the extensions of the modal concept of possibility and necessity whereas degrees of belief and plausibility are related to a language at a meta-level. Two models are considered: the transferable belief model and the possibility theory.

The Transferable Belief Model (TBM) is a model for quantified belief based on the use of belief functions. It corresponds to our interpretation of the Dempster-Shafer theory, and fits essentially with Shafer's initial proposal as described in his book, except for the following adaptations and explicitations.

1) Quantified beliefs are **point-valued**, not interval-valued
2) Any **connection with randomization or necessary additivity** as encountered within probability theory has been explicitly eliminated.
3) A difference has been established between **open and closed world assumptions**. The **normalization** after conditioning and combination is not performed in the open world context.
4) A **two-level model** for belief has been proposed. It consists of a **credal level** where belief is entertained and a **pignistic leve**l where belief is used to make decisions. At the credal level, belief is quantified by belief functions. At the pignistic level, beliefs are quantified by probability functions. When a decision must be made, the beliefs at the credal level are transformed into beliefs at the pignistic level, i.e. there exists a transformation from the belief functions to the probability functions. It is called the **pignistic transformation**. It corresponds to the Generalized Insufficient Reason Principle.
5) The justification of the TBM is based on the idea that the impact of an evidence consists in allocating parts m(A) of an initial unitary amount of belief among the

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propositions A of a given algebra. m(A) is that part of our belief that supports A and that, due to lack of information, does not support any strict subproposition of A.
6) The definition of bel (and pl) are derived from the basic belief masses, and the Choquet's inequalities characteristic of the belief functions are deduced.

Possibility theory is a model that aims in quantifying degrees of possibility and by duality, degrees of necessity. Its domains of application are either crisp propositions or fuzzy propositions.

Confusion has appeared because possibility functions turn out to be mathematically equivalent to consonant plausibility functions, the dual of the consonant belief functions. This mathematical analogy has lead to an erroneous assimilation of the two theories.

It will be argued that possibility theory deals with objective-physical problems and that the degree of possibility exists without any reference to human thinking. On the other part degrees of belief are related to subjective-personal opinions and result from the limitation of the human knowledge.

2) An example to show the difference.

As an example to illustrate the difference, let us consider that the universe of discourse Ω is a set of integers {0, 1, 2…} where integers represent the possible age X of somebody (e.g. John).

	no constraint (case 1)				X∈ A = [20, 50] (case 2)			
B	N	Π	bel	pl	N	Π	bel	pl
[150, 200]	0	0	0	0	0	0	0	0
[60, 90]	0	1	0	1	0	0	0	0
[30, 60]	0	1	0	1	0	1	0	1
[10, 60]	0	1	0	1	1	1	1	1
[0, 300]	1	1	1	1	1	1	1	1

Table 1: Degrees of necessity and possibility, belief and plausibility of various crisp sets B in case 1 (no constraint) and in case 2 where A is known to be true with A = [20, 50]

Case 1:

1.1) You have no further information. The left part of table 1 presents both the necessity N(B) and possibility Π(B) for various crisp sets B. Suppose 120 is the largest possible age. Let A = [0,120]. One has N(B)=1 if $A \subseteq B$, 0 otherwise, and Π(B) = 1 if $A \cap B \neq \emptyset$, 0 otherwise. N and Π reflect just a set relation between B and A. Everything up to here concerns the object-level of our discourse.

1.2.) Consider now that you have some prior beliefs about which value of Ω could be John's age. We thus have some bleif at the meta-level. In such a case you could build for every set $B\subseteq\Omega$ your degrees of belief and plausibility that John's age belongs to B.

1.3.) A particular case occurs when you ignore who is John and you are totally ignorant about which subset contains John's age. Total ignorance can be represented by a vacuous belief function Bel_0. Table 1 presents in columns 4 and 5 the degrees of belief and plausibility of the subset B when the a priori belief is vacuous. The results are numerically equal to the degrees of necessity and possibility, which does not mean that they correspond to the same concept. It just happens when the prior belief is vacuous.

Case 2: (right hand side of table 1)

2.1.) Suppose you learn that $X\subseteq A = [20, 50]$. The various degrees of necessity and possibility are computed as in case 1.

2.2.) Should one have a non-vacuous a priori belief bel_1 on Ω, then the information A induces a conditioning of bel_1 into $bel_{1|A}$ (by Dempster's rule of conditioning) and one could compute the degree of belief of any subset B of A as $bel_{1|A}(B)$.

2.3.) Should the prior belief be vacuous, then the degree of belief and plausibility computed are conditioning this vacuous belief function on A are numerically equal to the degrees of necessity and plausibility already computed.

Case 3:

3.1.) Suppose that you learn that John is YOUNG. Then $X\subseteq\tilde{A}$ where $\tilde{A}$ = "YOUNG" is a fuzzy subset of Ω whose membership function is $\mu_{\tilde{A}}(x)$ (the tilde indicates that the set is fuzzy). Given John is YOUNG, $\pi_{\tilde{A}}(x)$ is the possibility that John's age is x. Then for any crisp subset $B\subseteq\Omega$, $\Pi(B|\tilde{A}) = \max_{x\in B} \pi_{\tilde{A}}(x)$ and $N(B|\tilde{A}) = 1 - \Pi(\neg B|\tilde{A})$.

3.2.) Consider that you had a prior belief on Ω. The degrees of belief and plausibility of B given $\tilde{A}$ are in fact fuzzy degrees of belief and plausibility. Consider each α cut A_α where $A_\alpha = \{x: \mu_{\tilde{A}}(x) \geq \alpha\}$ for $\alpha\in(0,1]$. For each α, one can compute $bel(B|A_\alpha)$ by Dempster's rule of conditioning. Then $\widetilde{bel}(B|\tilde{A})$ is a fuzzy number characterized by the pairs $\{(bel(B|A_\alpha), \alpha): \alpha\in(0,1]\}$.

3.3.) Should your prior belief be vacuous then:

$$bel(B|A_\alpha) = \begin{cases} 1 & \text{if } A_\alpha\in B \\ 0 & \text{otherwise} \end{cases} \qquad pl(B|A_\alpha) = \begin{cases} 1 & \text{if } B\cap A_\alpha \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

If $A_1\not\subseteq B$, then $\forall\alpha$ $bel(B|A_\alpha) = 0$, and $pl(B|A_\alpha) = 1$ if $\alpha\leq\Pi(B|\tilde{A})$, 0 otherwise.

If $A_1\subseteq B$, then $bel(B|A_\alpha) = 1$ if $\alpha>\Pi(\neg B|\tilde{A})$, 0 otherwise, and $\forall\alpha$ $pl(B|A_\alpha) = 1$.

Suppose one proposes to define the crisp belief (plausibility) as the expected value of the fuzzy belief (plausibility):

$$pl(B|\tilde{A}) = \int_0^1 pl(B|A_\alpha)\, d\alpha = \Pi(B|\tilde{A})$$

$$bel(B|\tilde{A}) = \int_0^1 bel(B|A_\alpha)\, d\alpha = 1 - \Pi(\neg B|\tilde{A}) = N(B|\tilde{A}).$$

The same results are obtained when using the Sugeno integrals.

$$pl(B|\tilde{A}) = \max_\alpha pl(B|A_\alpha) \wedge \alpha = \Pi(B|\tilde{A})$$

$$bel(B|\tilde{A}) = 1 - \Pi(\neg B|\tilde{A}) = N(B|\tilde{A}).$$

Which definition is appropriate, if any, is an open question. Should such definitions of the crisp belief or plausibility be acceptable, then one has derived a tool to link consonant beliefs and fuzzy sets. Suppose the consonant belief function bel_1 on Ω. Build a fuzzy set $\tilde{A}$ on Ω such that our crisp belief $bel(B|\tilde{A})$ for any $B \subseteq \Omega$ induced by the knowledge $\tilde{A}$ and a vacuous prior belief function on Ω can be equal to $bel_1(B)$. Both contexts (bel_1 related to the consonant prior of Ω) and $bel(.|\tilde{A})$ (related to a fuzzy conditioning and a vacuous prior belief on Ω) lead to the same result for bel, therefore to the same state of belief (and to identical decisions once decision is involved). So we have been able to relate a consonant belief to a fuzzy set. Extension to non-consonant beliefs needs further studies.