# La théorie des possibilités quantitatives épistémiques vue comme un modèle de croyances transférables très prudent.

# Quantified Epistemic Possibilty Theory seens as an Hyper Cautious Transferable Belief Model.

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## Abstract

We provide a semantic for the values given to possibility measures. It is based on the semantic of the transferable belief model, itself based on the same approach as used for subjective probabilities. Besides we explain how the conjunctive combination of two possibility measures corresponds to the hyper-cautious conjunctive combination of the belief functions induced by the possibility measures.

## 1 Introduction

Quantitative possibility theory has been proposed as a numerical model which could represent quantified uncertainty (Zadeh, 1978; Dubois & Prade, 1998). It competes somehow with the probabilistic model (in its personalistic or Bayesian forms) and the transferable belief model (TBM) (Smets & Kennes, 1994; Smets, 1997, 1998b), both of which also intend to represent degrees of belief. A major issue when developing models to represent a psychological quantities, and belief is such an object, is to produce an operational definition of what the degrees are supposed to quantify. Such an operational definition, and the assessment methods that can be derived from it, provides a meaning, a semantic, to the .7 encountered in statements like 'my degree of belief is .7'. Such an operational definition has been produced long ago by the Bayesians (see Section 6.1) and recently for the TBM (see Section 6.3). So the numerical values encountered in these two models are well defined.

Quantitative epistemic possibility theory (QEPT) did not have such an operational definition, what lead to criticisms. One way to avoid them was to develop a qualitative epistemic possibility theory were only order relations are used (Dubois & Prade, 1998). Nevertheless QEPT seems a theory worth exploring, and rejecting it because of the lack of semantic would be unfortunate. Finding such a semantic would solve the problem, and this paper is just doing that.

For long, it had been realized that possibility functions are mathematically identical to consonant plausibility functions (Shafer, 1976), so using the TBM semantic to produce a QEPT semantic was an obvious attitude, even if left unjustified.

Suppose You (hereafter You is the agent who holds the beliefs) consider what beliefs You should adopt on what is the actual value of a variable $\Omega$. You have decided that Your beliefs should be those produced by a fully reliable source. Should You know the source's beliefs, they would be Yours. Unfortunately, it happens You only know the value of the 'pignisitc' probabilities the source would use to bet on the actual value of $\Omega$ (Smets, 1990; Smets & Kennes, 1994). The knowledge of the values of the probabilities allocated to the elements of $\Omega$ is not sufficient to construct the unique underlying belief function. Many belief functions can induce these probabilities. So all You know about the belief function that represents the source's beliefs is that it belongs to the set of beliefs that induce the collected pignistic probabilities. Obeying to

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a 'least commitment principle' that states that You should never give more beliefs than justified, You can select in that family the 'least committed' element. It happens it correspond to consonant plausibility function, hence to a possibility function. So a possibility function is the least committed belief function which pignistic transformation is equal to the pignistic probabilities collected from the source. This link had already been realized long ago. What was missing was showing that the analogy goes further.

Indeed in possibility theory, there exist a combination rule to conjunct two possibility functions. Let $\Pi_1$ and $\Pi_2$ be two possibility functions on $\Omega$. The most classical conjunctive combination rule to build $\Pi_{12}$ consists in using the minimum rule: $\Pi_{12}(\omega) = \min(\Pi_1(\omega), \Pi_2(\omega))$ for all $\omega \in \Omega$ and $\Pi_{12}(A) = \max_{\omega \in A \subseteq \Omega} \Pi_{12}(\omega)$.

It is well known that Dempster's rule of combination applied to two consonant plausibility functions does not produce a consonant plausibility function. So a blind application of Dempster's rule of combination on two possibility functions was not appropriate, and the analogy between consonant plausibility functions and possibility functions seems to collapse there.

In fact the solution comes from the fact Dempster's rule of combination was not the appropriate rule of conjunctive combination. Dempster's rule of combination requires that the involved pieces of evidence are 'distinct' and this property does not have to be satisfied in our present combination problem. Other rules exist based on some kind of cautious approach and where 'correlations' between the involved belief functions are considered.

Suppose You build two consonant plausibility functions, i.e., two possibility functions, using the method just presented. How to combine them conjunctively, not knowing if 'distinctness' is applicable. All You know it that the result of the combination must be a specialization of each of them (see Section 3.1). So consider all belief functions that are specialization of the two initial possibility functions. In that family, apply again the 'cautious' approach and select as Your belief the least committed element of that family. The result is a new consonant plausibility function and it turns out to be exactly the result one obtains within possibility theory when using the minimum rule. So the direct approach developed in possibility theory and the one derived using the TBM detour are the same. This result restores the coherence between the two models, and thus using the TBM operational definition to explain the meaning of the possibility values is perfectly valid and appropriate.

Therefore, QEPT is in fact a very cautious application of the TBM. It can use the operational definition of the TBM as an operational definition of the values of the possibility function. The link between possibility functions and grade of membership has been clarified by Zadeh (1978) and thus can be used directly here. We thus provide a semantic for both quantitative epistemic possibility theory and for fuzzy set theory.

In this paper we successively present the background material on the TBM (Section 2). Then we explain the concepts of specializations (Section 3), cautious combinations (Section 4) and pignistic transformation (Section 5). Then we explain the concepts of an operational definition both in probability theory and in the TBM (Section 6). Finally we show that possibility theory is indeed a very cautious TBM (Section 7). This implies that the semantic developed within the TBM can be applied to the semantic of the possibility measure, and automatically to the semantic of the grade of membership encountered in fuzzy set theory. An example illustrates how our method can be used (Section 7.4). Proofs can be found in the long version of this paper.

# 2 The transferable belief model

## 2.1 The transferable belief model

The TBM is a model for the representation of quantified beliefs held by a agent denoted You. The beliefs concern the value of the actual world, denoted $\omega_0$, which $\omega_0 \in /Omega$, the set of possible worlds, or equivalently the actual value of a variable $\Omega$.

The TBM is based on the assumption that beliefs manifest themselves at two mental levels: the credal level where beliefs are entertained and the pignistic level where beliefs are used to make decisions. At he credal level, beliefs are represented by belief functions whereas at the pignistic level, the beliefs induce a probability function which values are used to compute the expected utilities needed in order to take optimal decisions. The transformation between the belief function held at the credal level and the pignistic probabilities used at the pignistic level is called the pignistic transformation (see Section 5).

The central element of the TBM is the basic belief assignment, denoted $m$. For $A \subseteq \Omega$, $m(A)$ is the part of Your belief that supports $A$ (i.e. $\omega_0 \in A$), and that, due to lack of information, does not support any strict subset of $A$. The *focal elements* of a belief function are the subsets of $\Omega$ whose basic belief masses are positive. If some further pieces of evidence become available to You and You accept them as valid, and if their only impact bearing on $\Omega$ is that they imply that the actual world $\omega_0$ does not belong to $\overline{B}$, then the mass $m(A)$ initially allocated to $A$ is transferred to $A \cap B$. Indeed, some of Your belief (quantified by $m(A)$) was allocated to $A$, and now You accept that $\omega_0 \notin \overline{B}$, so that mass $m(A)$ is transferred to $A \cap B$ (hence the name of the model). The resulting new basic belief assignment is the one obtained by the application of Dempster's rule of conditioning .

The degree of belief $bel(A)$ quantifies the total amount of justified specific support given to $A$. It is obtained by summing all basic belief masses given to subsets $X \subseteq A$ (and $X \neq \emptyset$). Indeed a part of belief that supports that the actual world $\omega_0$ is in $B$ also supports that $\omega_0$ is in $A$ whenever $B \subseteq A$. So for all $A \subseteq \Omega$,

$$bel(A) = \sum_{\emptyset \neq B \subseteq A} m(B)$$

The degree of plausibility $pl(A)$ for $A \subseteq \Omega$ quantifies the maximum amount of potential specific support that could be given to $A$. It is obtained by adding all the basic belief masses given to subsets $X$ compatible with $A$, i.e., such that $X \cap A \neq \emptyset$:

$$pl(A) = \sum_{B \cap A \neq \emptyset} m(B) = bel(\Omega) - bel(\overline{A}).$$

Other functions we use repeatedly and that are in one to one correspondence with any of $m$, $bel$ and $pl$ are the commonality function q with:

$$q : 2^{\Omega} \rightarrow [0, 1],\ q(A) = \sum_{B: A \subseteq B} m(B),\ \forall A \subseteq \Omega$$

## 2.2 Notation

In order to enhance the fact that we work with non-normalized belief functions ($m(\emptyset)$ can be positive), we use the notation $bel$ and $pl$, whereas Shafer uses the notation $Bel$ and $Pl$.

Besides we use the next conventions that we have found convenient, even though it might seems cumbersome in some cases. The full notation for $bel$ and its related functions is:

$$bel^{\Omega,\Re}_{Y,t}[EC_{Y,t}](\omega_0 \in A) = x.$$

It denotes that $x$ is the value of the degree of belief held by the agent $Y$ (abbreviation for You) at time $t$ that the actual world $\omega_0$ belongs to the set $A$ of worlds, where $A$ is a subset of the frame of discernment $\Omega$ and $A \in \Re$ where $\Re$ is a Boolean algebra of subsets of $\Omega$. The belief is based on the evidential corpus $EC_{Y,t}$ held by $Y$ at $t$, where $EC_{Y,t}$ represents all what agent $Y$ knows at $t$. In practice many indices can be omitted for simplicity sake. Here $\Re$ is $2^{\Omega}$, the power set of $\Omega$. '$\omega_0 \in A$' is denoted as '$A$'. $Y$, $t$ and/or $\Omega$ are omitted when the values of the missing elements are clearly defined from the context. So $bel^{\Omega}[E](A)$ or even $bel(A)$ are often used.

Note that $bel^{\Omega,\Re}_{Y,t}[EC_{Y,t}]$ (and its simplified forms) denotes the belief function, and can be understood as a finite vector of length $|\Re|$, which components are the values of $bel^{\Omega,\Re}_{Y,t}[EC_{Y,t}](A)$ for every $A \in \Re$.

In the above notation, $bel$ can be replaced by any of $m$, $pl$, $q$, etc... The indices should made it clear what the links are. So $m^{\Omega,\Re}_{Y,t}[EC_{Y,t}]$ and $pl^{\Omega,\Re}_{Y,t}[EC_{Y,t}]$ are the bba and the plausibility function related to $bel^{\Omega,\Re}_{Y,t}[EC_{Y,t}]$.

## 2.3 Consonant belief functions

The subsets $A_1, A_2 \ldots A_n$ of $\Omega$ are said to be *nested* if:

$$A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n$$

A belief function is said to be *consonant* iff its focal elements are nested (Shafer, 1976, pg 219). By extension, we will speak of consonant basic belief assignments, commonality functions, plausibility functions . . . .

**Theorem 2.1** ***Consonant belief functions.*** *(Shafer, Theorem 10.1, pg 220) Let $m$ be a bba on $\Omega$. Then the following assertions are all equivalent:*

1. *$m$ is consonant.*
2. *$bel(A \cap B) = \min(bel(A), bel(B)),\ \forall A, B \subseteq \Omega$.*
3. *$pl(A \cup B) = \max(pl(A), pl(B)),\ \forall A, B \subseteq \Omega$.*
4. *$pl(A) = \max_{\omega \in A} pl(\omega)$, for all non empty $A \subseteq \Omega$.*
5. *$q(A) = \min_{\omega \in A} q(\omega)$, for all non empty $A \subseteq \Omega$.*

Items 3 and 4 shows that consonant plausibility and belief functions are possibility and necessity functions, respectively. The fact that we work with unnormalized bba's does not affect these properties, being understood that we never require that possibility and necessity functions be normalized. The difference between $\Pi(\Omega)$ or $pl(\Omega)$ and 1, that equals $m(\emptyset)$) represents the amount of conflict between the pieces of evidence that were used to build these functions.

## 2.4 Conjunctive combinations

Let $E_1$ and $E_2$ be two pieces of evidence and let $m^{\Omega}[E_1]$ and $m^{\Omega}[E_2]$ be the bba's they induce on $\Omega$. Remember the symbols between [ and ] denote the pieces of evidence taken in consideration when building Your belief function. We want to build the bba that would result form the combination of the two pieces of evidence. There are two families of combination:

- the conjunctive combinations that build the bba given You accept that both sources are fully reliable.
- the disjunctive combinations that build the bba given You only accept that one source is fully reliable but You do not know which one.

Only the conjunctive case is considered here. The case of partially reliable source is not considered in this paper. It would result in introducing the concepts of discounting. Besides more complex combinations exist but they are not considered here.

### 2.5 Non interactive combinations

Suppose the two pieces of evidence are considered as 'distinct', an ill defined concept but some justifications can be found in (Shafer & Tversky, 1985; Smets, 1992). Mathematically, it means that the result of the combination is a function of $bel[E_1]$ and $bel[E_2]$ only. We denote the result of the combination by $m[E_1 \circledcirc E_2]$ and write $m[E_1 \circledcirc E_2] = m[E_1] \circledcirc m[E_2]$ where the $\circledcirc$ symbol is used to represent both the 'and' between the two pieces of evidence when they are distinct and the operator that maps the two bba's into a bba.
Then $\forall C \subseteq \Omega$ :

$$m[E_1 \circledcirc E_2](C) = \sum_{A\subseteq\Omega, B\subseteq\Omega, A\cap B=C} m[E_1](A)m[E_2](B)$$

in which case:

$$q[E_1 \circledcirc E_2](A) = q[E_1](A)q[E_2](A) \qquad \forall A \subseteq \Omega.$$

This rule correspond to Dempster's rule of combination, except for its normalization factor. We call it the conjunctive combination rule.

**Note on notation.** Historically, the conjunctive combination rule was the only rule introduced by Shafer who used the $\oplus$ symbol to denote it. But in order to cope with disjunctive rules, with correlated rules and interactive rules, we need extra symbols, and those proposed here seem to create some convenient coherence. Later on, we will introduce other operators that will be denoted by $\oslash$and $\otimes$ in order to denote various forms of cautious conjunctive combination rules.

# 3 Specialization

## 3.1 Specialization matrix

The concept of specialization is at the core of the transferable belief model (Klawonn & Smets, 1992). Let $m_Y^\Omega[BK]$ be the basic belief assignment that represents Your belief on $\Omega$ given the background knowledge $(BK)$ accumulated by You. The impact of a new piece of evidence $Ev$ induces a change in Your beliefs characterized by a redistribution of the basic belief masses of $m_Y^\Omega[BK]$ such that $m_Y^\Omega[BK](A)$ is distributed among the subsets of $A$. In an colloquial way, we would say that 'the masses flow down'
Let $s(B, A) \in [0, 1]$ be the proportion of the mass given to $A$ that flows into $B \subseteq A$ when You learn the new piece of evidence $Ev$. In order to conserve the whole mass given to $A$ after this transfer, the coefficients $s(B, A)$ must satisfy:

$$\sum_{B\subseteq\Omega} s(B, A) = 1 \qquad \forall A \subseteq \Omega$$

As masses can flow only to subsets, $s(B, A) = 0\ \forall B \not\subseteq A$. The matrix $\mathbf{S}$ of the coefficients $s(B, A)$ for $A, B \subseteq \Omega$ is called a specialization matrix on $\Omega$ (Moral, 1985; Yager, 1986; Dubois & Prade, 1986; Delgado & Moral, 1987; Kruse & Schwecke, 1990).
The constraints about the $s(B, A)$'s can also be justified by considering what would be the result of applying the specialization when $m_Y^\Omega[BK](A) = 1$. The result of the 'down flow' of the unique unitary mass is the bba $m$ with $m(B) = s(B, A)$ for all $B \subseteq \Omega$. So the sum of the $s(B, A)$ over $B$ must be 1. Furthermore $BK$ was such that You were sure that the actual world $\omega_0$ belongs to $A$, the result of the combination of $BK$ with $Ev$ must be such that the resulting beliefs must satisfy $pl(\overline{A}) = 0$. This just means that the bbm's are 0 for all $B \not\subseteq A$, what is just the property required by $s(B, A)$.
So we could write the column vector $s(., A)$ as $m_Y^\Omega[Ev, A]$, i.e., Your bba given the piece of evidence $Ev$ and the fact You know that the actual world belongs to $A$.
In order to simplify notation, we switch to the classical matrix notation. By convention the lines and columns of the matrices and the elements of the vectors are 'numerically' ordered as follow: $\emptyset$, $\{a\}$, $\{b\}$, $\{a, b\}$, $\{c\}$, $\{a, c\}$, $\{b, c\}$, $\{a, b, c\}$, $\{d\}$, $\{a, d\}$, etc ... The vectors whose components are the values of a basic belief assignment, belief function, plausibility function, commonality function function are vertical vectors denoted $m, bel, pl, q$, respectively.
Let $\mathbf{S}$ be the matrix which row $B$, column $A$ element is the coefficients $s(B, A)$. $\mathbf{S}$ could also be written as $\{m_Y^\Omega[Ev, A] : A \subseteq \Omega\}$. Beware that we do not say that $m_Y^\Omega[Ev, A]$ results from the conditioning of $m_Y^\Omega[Ev, \Omega]$ on $A$ as obtained with Dempster's rule of conditioning. The present revision process can be much more complex.
After learning $E_v$, the basic belief assignment $m_Y^\Omega[BK]$ is transformed into the new basic belief assignment

$m_Y^\Omega[BK, Ev]$ such that for all $B \subseteq \Omega$:

$$m_Y^\Omega[BK, Ev](B) = \sum_{A \subseteq \Omega} s(B, A) m_Y^\Omega[BK](A)$$

or in matrix notation:

$$m_Y^\Omega[BK, Ev] = \mathbf{S} \cdot m_Y^\Omega[BK].$$

The basic belief assignment $m_Y^\Omega[BK, Ev]$ is called a specialization of $m_Y^\Omega[BK]$. The 'and' between the $BK$ and $Ev$ is voluntarily represented by a comma, to distinguish it from the $\circledcirc$ used in the case where the two pieces of evidence are distinct.

**Note.** The dual of the specialization is the generalization that corresponds to a disjunctive combination where masses 'flow upward' (Klawonn & Smets, 1992). It is not used here.

## 3.2 Ordering belief functions

When we will study that the beliefs induced by the knowledge of some betting behavior of an agent, we will find that many belief functions could satisfy the known constraints. In that family, we will select the 'minimal' element as the one representing Your beliefs. The solution will be a possibility functions. To achieve that goal, we need to define an ordering on the set of belief functions and to explain why the 'minimal' solution is adequate, i.e., what is the 'principle of least commitment'.

Dubois and Prade (1987) have proposed three solutions to order belief functions according to the 'strength' of the beliefs they represent. The intuitive idea is that the smaller the focal elements, the stronger the beliefs. Let $m_1$ and $m_2$ be two bba's on $\Omega$. Their proposals are:

- *pl-ordering.* If $pl_1(A) \geq pl_2(A)$ for all $A \subseteq \Omega$, we write $m_1 \sqsubseteq_p lm_2$
- *q-ordering.* If $q_1(A) \geq q_2(A)$ for all $A \subseteq \Omega$, we write $m_1 \sqsubseteq_q m_2$
- *s-ordering.* If $m_1$ is a specialization of $m_2$, we write $m_1 \sqsubseteq_s m_2$

They prove that :

- $m_1 \sqsubseteq_s m_2$ implies $m_1 \sqsubseteq_p lm_2$ and $m_1 \sqsubseteq_q m_2$, but the reverse is not true, and
- $m_1 \sqsubseteq_p lm_2$ and $m_1 \sqsubseteq_q m_2$ do not imply each other.

The s-ordering is thus stronger then the other twos as it implies them. Whenever $m_1 \sqsubseteq_X m_2$ for $X \in \{s, pl, q\}$, we say that $m_1$ is X-less committed than $m_2$ or that $m_2$ is X-more committed than $m_1$. The same qualification is extended to the functions related to the bba's.

The concept of 'least commitment' permit the construction of a partial order $\sqsubseteq$ on the set of belief functions (Yager, 1986; Dubois & Prade, 1987).

In (Smets, 1983; Smets & Magrez, 1985), we study the 'information content' of a bba, denoted $InfC(m)$. If we require that $InfC(m_1 \circledcirc m_2)$ be a function strictly monotone in both $InfC(m_1)$ and $InfC(m_2)$, then we show that $InfC(m^\Omega)$ must satisfy (up to any strict monotone transformation):

$$InfC(m^\Omega) = -\sum_{A \subseteq \Omega} c(A) \log(q^\Omega(A))$$

where $c(A) \geq 0$. $InfC = 0$ if $m$ is the vacuous belief function ($m(\Omega) = 1$)and is always non negative. The choice of $c(A)$ is not yet settled. Nevertheless $InfC$ can be used for ordering belief functions. So if $InfC(m_1) \leq InfC(m_2)$, we will write $m_1 \sqsubseteq_{InfC} m_2$. We have:

- if $\sqsubseteq_q$ then $\sqsubseteq_{InfC}$.
- If $m_1 \sqsubseteq_{InfC} m_2$ for any assignment of the $c(A)$ coefficients, then $m_1 \sqsubseteq_q m_2$

When the appropriate choice of the $c(A)$ will be found and justified, we feel the ordering of the bba will be based on $InfC$. Meanwhile we know that $m_1 \sqsubseteq_s m_2$ implies $m_1 \sqsubseteq_q m_2$, which implies $m_1 \sqsubseteq_{InfC} m_2$.

## 3.3 The Principle of Minimal Commitment

Suppose the information available to You corresponds to a set of constraints on the belief function that should represent Your beliefs. For instance, suppose You fully trust Your friend John, and You have no opinion whatsoever about the actual value of $\omega_0 \in \Omega$. John tells You that he believes at level .6 that $\omega_0 \in A \subseteq \Omega$, but he tells You nothing else. What would be Your belief on $\Omega$? All You know is that it should allocate a belief .6 to $A$. So if $bel_Y$ is the belief function that represents Your belief on $\Omega$, all You know is that $bel_Y(A) = .6$. Let $\mathcal{B}_{.6}$ be the family of belief functions on $\Omega$ that satisfy the .6 constraint. Which belief function in $\mathcal{B}_{.6}$ will You select to represent Your beliefs? The idea is to choose the one that is somehow the least committed in $\mathcal{B}_{.6}$, what reflects the principle 'never allocate more beliefs than necessary'. In the present example, the s-least committed belief function satisfies $m_Y(A) = .6, m_Y(\Omega) = .4$, and, of course, it is also the pl- and the q-least committed solution.

The principle evokes here is called the **Principle of Minimal Commitment**. It is really at the core of the TBM, where degrees of belief are degrees of 'justified' supports.

The **Principle of Minimal Commitment** consists in selecting the least committed belief function in a set of equally justified belief functions. The principle formalizes the idea that one should never give more support than justified to any subset of $\Omega$. It satisfies a form of skepticism, of uncommitment, of conservatism in the allocation of Your belief. In its spirit, it is not far from what the probabilists try to achieve with the maximum entropy principle (Dubois & Prade, 1987; Hsia, 1991).

Which order should be used? It is obvious that the best candidate is the s-ordering, as it implies the others. But when there is no s-least committed solution, the q-ordering seems to be appropriate, in particular because of its link with the InfC-order and also because of the meaning of $q$.

**The meaning of $q(A)$.** Although what some authors states, the commonality function has a clear meaning. When $\Omega = \{x, y\}$ the difference $pl(x) - bel(x)$ has often been proposed as a measure of the uncertainty in $bel$. In fact $pl(x) - bel(x) = m(\{x, y\})$ and $m(\{x, y\})$, as well as $m(\Omega)$ in general, is the part of belief free to flow anywhere, totally uncommitted. So to consider $m(\Omega)$ as the measure of uncertainty seems reasonable. Suppose now we know that the actual world belongs to $A \subseteq \Omega$. Then $m[A](A)$ obtained by conditioning $m$ with Dempster's rule of conditioning becomes the 'conditional measure of uncertainty' in context $A$. It just happens that $m[A](A) = q(A)$, so the commonality function is the set of conditional measure of uncertainty, and the fact that a measure of information content turns out to be a function of the $q$'s becomes very natural.

### 3.4 Specialization and Dempster's rules

In (Klawonn & Smets, 1992), we show that the effects of both Dempster's rules can be obtained by specialization matrices, and that Dempster's rules can easily be justified with commutativity requirements.

### 3.5 Belief revision

The specialization is the most general form of belief revision. It is 'interactive' in that the result of the conjunctive combination is not just a function of the beliefs that would be induced by each piece of evidence individually.

The typical 'non interactive' revision is Dempster's rule of combination, a special form of specialization. But there are other forms of 'non interactive' belief revision, like the 'cautious' ones considered now. We need this concept as the result of the conjunctive combination of two possibility functions is equal to the one obtained by a cautious interactive conjunctive combination of the two corresponding consonant plausibility functions.

## 4 Cautious combinations

### 4.1 General idea

Let $m_1$ and $m_2$ be two bba's on $\Omega$ induced by the two pieces of evidence $E_1$ and $E_2$, respectively. We know that their conjunctive conjunction results in a new bba on $\Omega$ which must be a specialization of both $m_1$ and $m_2$. We do not assume that $E_1$ and $E_2$ are distinct, nor that the combination must be non interactive. Any specialization is allowed.

Let $\mathcal{SP}(m_1)$ and $\mathcal{SP}(m_2)$ be the set of specializations of $m_1$ and $m_2$, respectively. The result of the combination of $m_1$ and $m_2$ belongs then to $\mathcal{SP}(m_1) \cap \mathcal{SP}(m_2)$. That family is never empty as $m_1 \textcircled{\cap} m_2$ always belongs to it. As far as all You know about the combination of $m_1$ and $m_2$ is that the result must be in $\mathcal{SP}(m_1) \cap \mathcal{SP}(m_2)$, the hyper-cautious attitude would consist in accepting it as representing Your beliefs. Let $m_{1 \textcircled{\curlywedge} 2}$ denote this least committed element. So $m_{1 \textcircled{\curlywedge} 2} = \min\{m : m \in \mathcal{SP}(m_1) \cap \mathcal{SP}(m_2)$. Unfortunately the minimum does not always exist, but when $m_1$ and $m_2$ are both consonant the minimal element exists as shown below.

We call the last combination, the hyper-cautious conjunctive combination rule, denoted by $\textcircled{\curlywedge}$. In the special case where the specializations must be dempsterian, this combination is called the cautious conjunctive combination rule and denoted by $\owedge$.

### 4.2 The case of consonant belief functions

As far as possibility functions are consonant plausibility functions, we consider what would be $m_{1 \textcircled{\curlywedge} 2}$ if both $m_1$ and $m_2$ are consonant, and prove that $m_{1 \textcircled{\curlywedge} 2}$ is equal to the result found when conjunctively combining two possibility functions.

**Theorem 4.1** *Let $m_1$ and $m_2$ be two consonant belief functions on $\Omega$ with $q_1$ and $q_2$ their corresponding commonality functions. Let $\mathcal{SP}_1$ and $\mathcal{SP}_2$ be the set of specializations of $m_1$ and $m_2$, respectively. Let $q_{12}(A) = \min(q_1(A), q_2(A))$ for all $A \subseteq \Omega$, and $m_{12}$ its corresponding bba. Then $m_{12} = m_{1 \textcircled{\curlywedge} 2} = \min\{m : m \in \mathcal{SP}(m_1) \cap \mathcal{SP}(m_2)$.*

Note that if $\Pi_1$ and $\Pi_2$ are two possibility functions with $q_1$ and $q_2$ their related commonality functions, the commonality function $q_{12}$ of their conjunctive combination $\Pi_{12}$ satisfies : $q_{12}(A) = \min(q_1(A), q_2(A))$ for all $A \subseteq \Omega$.

# 5 The pignistic probability function for decision making

Suppose a bba $m^\Omega$ that quantifies Your beliefs on $\Omega$. When a decision must be made that depends on the actual value $\omega_0$ where $\omega_0 \in \Omega$, You must construct a probability function in order to take the optimal decision, i.e., the one that maximizes the expected utility. This is achieved by the pignistic transformation. Its nature and its justification are defined in (Smets, 1990; Smets & Kennes, 1994; Smets, 1998b).

Let $F$ be the *betting frame*, the set of 'atoms' on which stakes will be allocated. Bets can then only be built on the elements of the power set of that frame. Let $BetP^F$ denote the pignistic probability function You will use to bet of the alternatives in $F$. $BetP^F$ is a function of $F$ and $m^\Omega$,

$$BetP^F = \Gamma(m^\Omega, F).$$

We show that the only transformation from $m^\Omega$ to $BetP^F$ that satisfies some rationality requirements is the so called pignistic transformation that satisfies:

$$BetP^F(f) = \sum_{A: f \in A \subseteq F} \frac{m^F(A)}{|A|(1 - m^F(\emptyset))}, \qquad \forall f \in F \tag{1}$$

where $|A|$ is the number of elements of $F$ in $A$, and $m^F$ is the bba induced by $m^\Omega$ on $F$, (we have assumed that $F$ is compatible with $\Omega$ (Shafer, 1976, pg. 114 *et seq.*).

It is easy to show that the function $BetP^F$ is indeed a probability function and the pignistic transformation of a probability function is the probability function itself. We call it pignistic in order to avoid the confusion that would consist in interpreting $BetP^F$ as a measure representing Your beliefs on $F$.

# 6 Operational definitions of degrees of beliefs

Why do we need an operational definitions of degrees of beliefs? When I write $bel(A) = .67$, what means the .67. How can we give it a meaning that is somehow objective, and that we could share. Producing an operational definition of .67 (and all other values of course) means producing the description of a publicly observable experiment which would produce the value .67 when and only when the degree of belief is in fact .67. In probability theory, this is achieved by observing the betting behavior of the belief holder. The same method is used in the TBM. As far as we know, none had been produced and justified for QEPT, a gap we fill in this paper.

## 6.1 An operational definition of $P$.

The classical definition of a subjective probability is based on an analysis of rational betting behavior. The (subjective) probability of a proposition is usually characterized as the value of the opportunity to gain a unit value if the proposition is true (Ramsey, 1964). More formally, one variant of the operational definition of a subjective probability is the following:

**Definition 6.1** ***Operational definition of subjective probabilities.*** *Consider a finite space $\Omega$, a game on the betting frame $\Omega$, a player and a banker. We have '$P^\Omega_{You,t}(A) = x$' iff You consider at time $t$ and for any $M > 0$ that the player must pay \$$xM$ to the banker to enter a game where the player wins \$M from the banker if the actual world belongs to A and \$0 otherwise, and You are ready to be any of the player or the banker.*

We insist on the fact that You are not allowed to 'run away' from the game. You must accept to be either the banker or the player, this being settled after You have assessed the value of $x$. The present definition is based on 'forced bets'.

## 6.2 The assessment of $P$.

In order to assess the value of a subjective probability, one can consider the following method. Let a finite space $\Omega$ and $A \subseteq \Omega$. Consider two bets. In bet 1, You bet on $A$ versus $\overline{A}$ where You gain \$$M$ if $A$ is true, and \$0 otherwise (with \$$M$ being any reasonable prize like \$100). In bet 2, You have an urn with a proportion $p$ of Black balls. You bet on Black versus not-Black where You gain \$$M$ if the randomly selected ball (where every ball has the same chance to be selected) is Black, and \$0 otherwise. Which bet do You prefer?

- If You prefer bet 1, it means that $P(A) > p$.
- If You prefer bet 2, it means that $P(A) < p$.
- If You are indifferent between the two bets, it means that $P(A) = p$.

By varying $p$, one can (in theory) always find a state of indifference between the two bets. So one can assess the value of $P(A)$.

In practice, this method is too crude to assess probabilities and more elaborated methods have been developed by psychometricians. Nevertheless many of the methods they developed are ingenious variants of the one we just described.

### 6.3 An operational definition of *bel*.

The pignistic transformation can be used in order to provide both an operational definition of the degrees of belief, and a method to assess them. The approach is essentially identical to the one encountered in subjective probability theory except we use the possibility to construct several betting frames (see section 6.1).

**Definition 6.2** ***Operational definition of degrees of belief.*** *Suppose a finite space $\Omega$, a family of games $G = \{G_1, G_2 \dots\}$ built on the betting frames $F_i, i = 1, 2 \dots$, respectively, where each frame is compatible with $\Omega$. Suppose a player and a banker. Consider one game $G_i \in G$ and its betting frame $F_i$. Suppose $A$ is discerned by $F_i$. We have $'BetP^{F_i}_{You,t}(A) = x'$ iff You consider at time $t$ that the player must pay \$$xM$ to the banker to enter the game $G_i$ where the player wins \$M from the banker if the actual world belongs to $A$ and \$0 otherwise, and You are ready to be any of the player or the banker. Consider then all possible games $G_i$ on $G$. Then $bel^{\Omega}_{You,t}$ is the belief function on $\Omega$ such that $BetP^{F_i}_{You,t} = \Gamma(bel^{\Omega}_{You,t}, F_i)$, $\forall i = 1, 2.....$*

Given the finiteness of $\Omega$, there is always a finite number of betting frames $F_i$ which will be sufficient to derive uniquely $BetP^{\Omega}_{You,t}$, so the definition is realizable.

It is important to realize that the pignistic probability functions obtained with different frames are not necessarily related between them by the laws of probability. So you could bet on $A$ versus $B$ where $B = \overline{A}$ with pignistic probabilities of 1/2 and 1/2, and on $A$ versus $B_1$ versus $B_2$ where $B_1 \cap B_2 = \emptyset$ and $B_1 \cup B_2 = B$ with pignistic probabilities of 1/3, 1/3 and 1/3 (this is encountered in case of total ignorance on $\Omega = A \cup B$ (Smets & Kennes, 1994).

### 6.4 The assessment of *bel*.

In (Smets, 1998a), we explain in detail and illustrate how the bba's can be assessed. Here, we present only the general procedure.

The assessment of a belief function is essentially obtained through a schema based on preference between gambles (see section 6.2).

The method proposed in probability theory extends directly to belief functions. It is based on using several betting frames. Let a finite set $\Omega$ and a family of compatible betting frames $F_1$, $F_2$... For each $F_i$, we assess $BetP^{F_i}$ using the preference ordering between two bets as done in section 6.2. We then determine the set $BF_i^{\Omega}$ of belief function on $\Omega$ which pignistic transformation on $F_i$ is $BetP^{F_i}$. We repeat the procedure with each $F_i$'s. Then $bel^{\Omega}$ belongs to the intersection of all the $BF_i^{\Omega}$. If the intersection is empty, then it means the pignistic probability functions are inconsistent, what ideally should not occur, but it happens of course in practice, just as in probability theory where people assess probabilities that usually violate Kolmogorof axioms. It essentially translates the imprecision of the assessment tool. Thanks to the fact that a belief function is defined by a finite number of values and the possibility to build as many betting frames as one needs, the intersection can be such that it contains only one belief function.

## 7 From the TBM to QEPT

### 7.1 The q-least committed belief function induced by the knowledge of the pignistic probabilities on the singletons of $\Omega$

Suppose a source $S$ (an agent) held some beliefs over $\Omega$, and You would adopt these beliefs if You come to know them. It happens that the only information available to You about these beliefs are the pignistic probabilities that $S$ would use to bet on the singletons of $\Omega$, denoted $BetP^{\Omega}_S$. So all You know it that the belief function that represents $S$'s beliefs, denoted $bel^{\Omega}_S$, belongs to the set of belief functions that share the same pignistic probabilities on $\Omega$ (we call this set the set of 'isopignistic' belief functions). Not knowing which belief function in that set is the one held by the agent, and being cautious, Your beliefs should be represented somehow by the least committed element of that set of belief functions.

We successively define the set of isopignistic belief functions, show that its q-least committed element is unique and is a consonant belief function, hence a possibility function.

**Definition 7.1** ***The set of isopignistic belief functions.*** *Let $m^{\Omega}_0$ be a bba defined on $\Omega$ and $BetP^{\Omega}_0 = \Gamma(m^{\Omega}_0, \Omega)$ be its related pignistic probability function. The set of belief functions on $\Omega$ which pignistic transformation equals $BetP^{\Omega}_0$ is called the set of isopignistic belief functions induced by $BetP^{\Omega}_0$ and denoted $\mathfrak{B}isoP(BetP^{\Omega}_0)$. More formaly:*

$$\mathfrak{B}isoP(BetP^{\Omega}) = \{m^{\Omega} : \Gamma(m^{\Omega}, \Omega) = BetP^{\Omega}\}.$$

**Theorem 7.1** *Let $BetP^{\Omega}$ be a pignistic probability function defined on $\Omega$ with the elements $\omega_i$ of $\Omega$ so* *labeled that :*

$$BetP^\Omega(\omega_1) \geq BetP^\Omega(\omega_2) \geq \ldots \geq BetP^\Omega(\omega_n)$$

*where* $n = |\Omega|$. *Let* $\mathfrak{B}isoP(BetP^\Omega)$ *be the set of isopignistic belief functions induced by* $BetP^\Omega$. *The q-least committed bba in* $\mathfrak{B}isoP(BetP^\Omega)$ *is the consonant bba* $\widehat{m}$ *which non zero bbm are, with* $A = \{\omega_1, \omega_2 \ldots \omega_i\}$*:*

$$\widehat{m}(A) = |A| \cdot (BetP^\Omega(\omega_i) - BetP^\Omega(\omega_{i+1}))$$

*where* $BetP^\Omega(\omega_{n+1})$ *is 0 by definition.*

Therefore we know what is the q-least committed bba among all the bba's that share the same pignistic probabilities. It is unique and corresponds to the bba related to a consonant plausibility function, thus a possibility function. Nevertheless this element is neither the s- nor the pl-least committed element, as these do not necessarily exist.

## 7.2 Conjunctive combinations

Suppose all You know about the value of some actual world in $\Omega$ are the pignistic probabilities on $\Omega$ held by two agents, denoted $BetP_1^\Omega$ and $BetP_2^\Omega$, respectively. From each pignistic probability function, You build the q-least committed belief function that represent Your beliefs according to the information You collected from the corresponding agents. Then You conjunctively combined these two belief functions. All You know is that the result of the combination must be a specialization of the two belief functions You had built. Being 'hyper' cautious, You will select in the set of possible solutions, the s-least committed one. As shown in Section 4.2, the solution is unique and well defined.
If $q_1$ and $q_2$ ere the commonality functions derived from Your knowledge of the pignistic probabilities of the two agents, then the commonality function of the combination satisfies:

$$q_{1\textcircled{\scriptsize$\wedge$}2}(A) = \min(q_1(A), q_2(A)) \; \forall A \subseteq \Omega.$$

This commonality function is consonant, and thus correspond to the commonality function related to a possibility function.
This is exactly the possibility function that is obtained within quantitative epistemic possibility theory (QEPT).
So QEPT corresponds to a hyper cautious usage of the TBM theory. The importance of the result is that the assessment of the numerical values of a possibility function is realized by the same method of described in Section 6.4. Our development thus produces an operational definition of the values of the possibility function used in QEPT, providing thus an answer to a classical criticism that claims that QEPT lacks a semantic, i.e., that the numerical values allocated to degrees of possibilities seems more or less arbitrary. We just show that they correspond to the least committed solution induced by the knowledge of the betting behavior on the elements of $\Omega$. This situation seems quite classical, and should often been encountered. Getting only the betting behavior on $\Omega$ seems quite natural, people will hardly tell You spontaneously how they would bet on several frames. So it seems natural to expect the present situation as very common. This might explain the interest of QEPT.

## 7.3 The possibility functions induced by two fuzzy constraints

We present now an example where we use the whole procedure developed so far. Suppose $\Omega$ is a set of n individuals, denoted $\omega_1, \ldots \omega_n$. One of them has been selected (I have not said randomly selected, I just say 'selected') and put in a given room, denoted R.
There are two agents, John and Tom. John knows the height of each individuals, and Tom knows their weights. A third agent looks at the selected individual and states to John 'he is tall' and to Tom 'he is obese', these two terms being of course 'fuzzy'. John and Tom communicate to You what are their personal opinions about who is the individual in room R.
According to possibility theory, each of John and Tom would produce a possibility distribution on $\Omega$ given 'tall' and given 'obese', respectively. Let us denote them $\pi^\Omega[tall]$ and $\pi^\Omega[obese]$. Then You would combine these two possibility distributions with the conjunctive rule into $\pi^\Omega[tall\ and\ obese](\omega_i) = \min(\pi^\Omega[tall](\omega_i), \pi^\Omega[obese](\omega_i))$ for $i = 1 \ldots n$. This is exactly what we get if we take the TBM detour. The only problem with the present approach is that the 'meaning' of the values of the two initial possibility distributions is unclear (they lack a semantic).
In the TBM, You first consider John. You ask him how he would bet on which of the individual of $\Omega$ is in the R room, now that he knows he is tall. Using the method developed in Section 6.2, You generate the probabilities John would use to bet on $\Omega$. Suppose You can only obtain these pignistic probabilities for the singletons of $\Omega$ and cannot run the whole assessment procedure described in Section 6.4. So all You know from John is $BetP_J^\Omega$. From it You build the underlying q-least committed belief function, that is a consonant plausibility function (a possibility function) as shown in Section 7.1. You do the same with Tom, once he knows the individual in R-room is obese. So You end up with two consonant plausibility functions.

| $\Omega$ | height | weight | $BetP[Tall]$ | $BetP[Obese]$ |
|---|---|---|---|---|
| a | 185 | 95 | .4 | .3 |
| b | 190 | 90 | .5 | .2 |
| c | 160 | 110 | .1 | .5 |

Table 1: Height and weight of three individuals, and the pignistic probabilities given by John (on height) and Tom (on height) when they know only that the selected individual is '*Tall*' and '*Obese*', respectively.

| $2^\Omega$ | $m[T]$ | $\Pi[T]$ | $q[T]$ | $m[O]$ | $\Pi[O]$ | $q[O]$ |
|---|---|---|---|---|---|---|
| $\emptyset$ | | 0 | 1 | | 0 | 1 |
| {a} | | .9 | .9 | | .8 | .8 |
| {b} | .1 | 1. | 1. | | .6 | .6 |
| {a,b} | .6 | 1. | .9 | | .8 | .6 |
| {c} | | .3 | .3 | .3 | 1. | 1. |
| {a,c} | | .9 | .3 | .2 | 1. | .8 |
| {b,c} | | 1. | .3 | | 1. | .6 |
| {a,b,c} | .3 | 1. | .3 | .6 | 1. | .6 |

Table 2: Values of the bba's and their related $\Pi = pl$ and $q$ functions derived from the two pignistic probability functions of Table 1

| $2^\Omega$ | $q[T] \wedge q[0]$ | $m[T] \otimes m[O]$ | $\Pi[T \wedge O]$ |
|---|---|---|---|
| $\emptyset$ | 1. | .2 | 0 |
| {a} | .8 | .2 | .8 |
| {b} | .6 | | .6 |
| {a,b} | .6 | .2 | .8 |
| {c} | .3 | | .3 |
| {a,c} | .3 | | .8 |
| {b,c} | .3 | | .6 |
| {a,b,c} | .3 | .3 | .8 |

Table 3: Construction of the very cautious combination of the two bba's of Table 2 and its related $m$ and $\Pi$ function.

| $\Omega$ | height | weight | $\mu_T(.)$ | $\mu_O(.)$ | $\mu_{T\ \&\ O}(.)$ |
|---|---|---|---|---|---|
| a | 185 | 95 | .9 | .8 | .8 |
| b | 190 | 90 | 1. | .6 | .6 |
| c | 160 | 110 | .3 | 1. | .3 |

Table 4: Height and weight of the three individuals of Table 1, and the grades of membership of each individual in the sets of '*Tall*' men, of '*Obese*' men and of '*Tall & Obese*' men. Note the values in the last column can be derived from the two on its left or as well from the $\Pi$ column in Table 3

Now You combine them, but being ignorant about the relation between 'tall' and 'obese', and about any relation that might exist between what John and Tom said, You apply the hyper cautious combination rule and build $m_Y^\Omega[Tall,\ Obese] = m_J^\Omega[Tall] \otimes m_T^\Omega[Obese]$ as described in Section 7.2. The result is a belief function which is the same as the one obtained by the conjunctive combination of the two possibility functions derived from the pignistic probabilities. But at least, thanks to the TBM detour, we have been able to justify the origin of the values used for the two possibility functions.

### 7.4 A numerical example

The next numerical example illustrates the procedure followed through the TBM detour. Let $\Omega = \{a, b, c\}$. Tables 1 to 4 summarize most numerical results.

## 8 Conclusions

A major criticism addressed at quantitative epistemic possibility theory (QEPT) concerns its lack of semantic, i.e., the meaning of the numbers used to represent the degrees of possibility are not justified by an operational definition. Such a definition consists in describing a publicly observable experiment where the value of the possibility could be measured in a non ad hoc way. Such definitions exist for subjective probability theory and for the transferable belief model (TBM). We show that the operational definition developed for the TBM can be used as well for QEPT, providing it with the requested operational definition.

This definition is based on the idea that the user has only been able to collect the probabilities used to bet on the elements of the frame $\Omega$ on which the possibility must be assessed. From these probabilities, You build the least committed plausibility function compatible with the collected probabilities. The least commitment principle that underlies this choice just states that You should never allocate more beliefs than justified by what You know. The least committed plausibility function happens to be consonant, hence it is a possibility function. We thus propose to define the possibility value as the one derived by the previous approach based on the knowledge of the pignistic probabilities on $\Omega$ and the least commitment principle.

To be satisfactory, the solution must also satisfied the major combination rule. In possibility theory the conjunction is obtained by applying the minimum rule on the possibility distribution functions. In the TBM detour, it consists in combining the two consonant plausibility functions by the hyper cautious interactive conjunction combination, i.e., considering that the result of the combination must be the least committed specialization of the two consonant plausibility functions. We just prove that the result of the minimum rule applied in possibility theory is the one obtained by the hyper cautious conjunctive combination used in the TBM. If it had not been the case, the semantic of possibility theory based on the TBM would not have been adequate as the resulting functions would not share the same behavior once combination is considered.

In conclusion, we have produced a semantic for QEPT, hoping to solve the criticism addressed by those who require such a definition before using a model that without it seems too much ad hoc.

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