No Dutch Book can be built against the TBM even though update is not obtained by Bayes rule of conditioning.

Philippe SMETS[1]
IRIDIA, Université Libre de Bruxelles
50 av. F. Roosevelt, CP194/6. B-1050 Brussels, Belgium.

Summary: Synchronic and Diachronic Dutch Books are used to justify the use of probability measures to quantify the beliefs held by a rational agent. The argument has been used to reject any non-Bayesian representation of degrees of beliefs. We show that the transferable belief model resists the criticism even though it is not a Bayesian model. We analyze the 'Peter, Paul and Mary' example and show how it resists to Dutch Books.

1. Introduction.

1.1. Overview

Some Bayesians used Dutch Book arguments in order to criticize any quantified representation of belief that do not satisfy the Bayesian model, i.e. beliefs are quantified by probability functions, and their update is obtained by the application of Bayes rule of conditioning. Any agent whose beliefs do not satisfy this representation is incoherent in that a Bookie could always build a set of bets such that the agent would loose for sure. Synchronic and Diachronic Dutch Books have been defined to justify the static representation and the dynamic behavior of the agent's beliefs, respectively (Teller, 1973, Jeffrey, 1988).

We show that the transferable belief model (TBM) and its pignistic transformation resist such criticisms, and that the Dutch Book argument cannot be used to criticize the TBM. In the TBM, we accept that decisions are based on maximizing the expected utility, hence they are necessarily based on some probability measures and utility functions (Savage, 1954). What we reject is that beliefs are represented by probability measures, and that the updating of the agent beliefs comply with Bayes rule of conditioning.

---

[1] This work has been partially funded by the CEC-ESPRIT III Basic Research Project 6156 (DRUMS II), and the Communauté Française de Belgique, ARC 92/97-160 (BELON).

We summarize the TBM, a model for representing quantified beliefs based on the belief functions, and its related pignistic transformation, that induced the probability measure to be used for optimal decisions making. Synchronic Dutch Books are avoided as all decisions are based on so called pignistic probabilities, i.e. bona fide probability functions on the set of alternative states of nature. The avoidance of the Diachronic Dutch Book is much more delicate to show and the major part of this paper is devoted to this topic. The reason we are immune to Dutch Books resides in the non acceptation of the Temporal Coherence Principle that somehow underlies the Diachronic Dutch Books. We do not assume that our betting behaviour is the same for hypothetical bets (before the conditioning event has occurred) and for factual bets (once the conditioning event has occurred). With hypothetical bets, all possible conditioning events can be considered whereas with factual bets, only the event that actually occurred can be used as a conditioning event. The distinction is quite similar to the one between conditional and a posteriori probabilities as described in Goldstein (1985).

1.1. The Transferable Belief Model.

The transferable belief model is a model for representing the quantified beliefs held by an agent (You) at a given time on a given frame of discernment. It concerns the same concept as those considered by the Bayesian model except it does not rely on probabilistic quantification but on belief functions.

The **transferable belief model** is based on:

- a two-level model: there is a *credal level* where beliefs are entertained and a *pignistic level* where beliefs are used to make decisions (from *pignus* = a bet in Latin, Smith 1961).
- at the credal level beliefs are quantified by belief functions.
- the credal level precedes the pignistic level in that, at any time, beliefs are entertained (and updated) at the credal level. The pignistic level appears only when a decision needs to be made.
- when a decision must be made, beliefs at the credal level induce a probability measure at the pignistic level, probability measure needed to compute the expected utilities. I.e., there is a *pignistic transformation* from belief functions to probability functions.
- when a new piece of evidence is accepted, the agent’s beliefs are updated into a new belief by the application of Dempster's rule of conditioning (see section 1.2.3).

Usually the two levels are not distinguished and probability functions are used to quantify degrees of belief at both levels. The justification is usually linked to "rational" agent behaviour within betting and decision contexts (DeGroot, 1970). The Bayesians have convincingly showed that if decisions must be "coherent", our belief over the various possible outcomes must induce a probability function that is used to compute the expected utilities that must be maximized. This result is accepted here, except that the *probability function does not quantify our beliefs, but is induced by them when a decision is really involved*. That beliefs are necessary ingredients for our decisions does not mean that beliefs cannot be entertained without any revealing behaviour manifestations (Smith and Jones, 1986, p.147).

We claim that beliefs can indeed be entertained without any concept of decision. For instances, I can entertain beliefs about meta-physical problems even though I am not going to make any decision about it. I can have some beliefs about the status of the red light down my street in Brussels even though I am not in Brussels for the moment and no decision will be made that depends on the color of the red light.

A full description of the models for beliefs representation based on belief functions can be found in Shafer's book (1976). A somehow revised version appears in Smets (1988). The transferable belief model is described in Smets and Kennes (1994) where we also describe the procedure for assessing degrees of belief, a procedure based on exchangeable bets established on different betting frames. The axiomatic justification of the use of belief functions to quantify beliefs is given in Smets (1993c). Justifications of the conditioning rule can be found in Klawonn and Smets (1992), in Nguyen and Smets (1993) and in Kruse, Nauck and Klawonn (1991) where differences between various updating concepts are considered. Further results on Bayes theorem and the disjunctive rule of combination appear in Smets (1978, 1993a). Jeffrey's rule of conditioning extended to the TBM are presented in Smets (1993b).

1.2. The Mathematics of the TBM.

1.2.1. The Frame of Discernment.

Let L be a finite **propositional language**, and $\Omega = \{\omega_1, \omega_2, ...\omega_n\}$ be the set of **worlds** that correspond to the interpretations of L. **Propositions** identify to subsets of Ω. Beliefs and probabilities given to propositions can thus be identically considered as beliefs and probabilities given to subsets of Ω. The set notation will be used hereafter. By definition there is an **actual world** ϖ and it is an element of Ω. For $A \subseteq \Omega$, bel(A) denote the degrees of belief that the actual world ϖ belongs to A. For simplicity sake, we admit that bel will be defined for every subsets of Ω, so bel are function from 2^Ω to [0,1]. Ω is called the **frame of discernment** .

1.2.2. The Basic Belief Masses.

Basic Assumption.
*The **TBM** postulates that the impact of a piece of evidence on an agent can be expressed by an allocation of parts of an initial unitary amount of belief among the subsets of Ω. For A⊆Ω, m(A) is a part of the agent's belief that supports A i.e. that the 'actual world ϖ is in A, and that, due to lack of information, does not support any strict subset of A.*

The m(A) values, A⊆Ω, are called the **basic belief masses** (bbm) and the m function is called the **basic belief assignment** .

Let m: $2^{\Omega} \rightarrow [0,1]$ with

$$\sum_{A \subseteq \Omega} m(A) = 1$$

The difference with probability models is that positive masses can be given to any subsets of Ω and not only to the elements of Ω.

As an example, let us consider a somehow reliable witness in a murder case who testifies to You that the killer is a male. Let α = .7 be the reliability You give to the testimony. It results from the probability .7 You have that the witness saw the killer. Suppose that *a priori* You have an equal belief that the killer is a male or a female. A classical probability analysis would compute the probability P(M) of M where M = 'the killer is a male'. P(M) = .85 = .7 + .5 x.3 (the probability .7 that the witness is reliable (saw the killer) in which case M is true for sure, plus the probability .5 of M given the witness is not reliable (did not see the killer) weighted by the probability .3 that the witness is not reliable (did not see the killer)). The TBM analysis will give a belief .7 to M. The .7 can be viewed as the *justified* component of the probability given to M (called the belief or the support) whereas the .15 can be viewed as the *contextual* component of that probability. The TBM deals only with the justified components.

1.2.3. Updating.

If some further evidence becomes available to You and implies that B is true, then the basic belief mass m(A) initially allocated to A is transferred to A∩B. Hence the name TBM (*transferable* belief model)

Continuing with the murder case, suppose there are only two potential male suspects: Phil and Tom. Then You learn that Phil is not the killer. The testimony now supports that the killer is Tom. The reliability .7 You gave to the testimony initially supported 'the killer is Phil or Tom'. The new information about Phil implies that .7 now supports 'the killer is Tom'.

The transfer of belief described in the TBM corresponds to the so-called **unnormalized Dempster rule of conditioning**. Let m be a basic belief assignment on the frame of discernment Ω and suppose the conditioning evidence tells You that the actual world is in $B \subseteq \Omega$, the basic belief assignment m are transformed into $m_B: 2^\Omega \to [0,1]$ with:

$$m_B(A) = \sum_{X \subseteq \overline{B}} m(A \cup X) \qquad \text{for } A \subseteq B$$

$$m_B(A) = 0 \qquad \text{for } A \not\subseteq B$$

Note that a non null basic belief mass could be given to Ø. Its meaning is analyzed in Smets (1992). m(Ø) corresponds to the amount of contradiction present in the basic belief assignment m, as encountered when two sources of information give some support to contradictory hypothesis. For any belief function bel, its normalization is obtained by dividing the bbm's by 1-m(Ø).

1.2.4. Belief Functions.

Given Ω, the **degree of belief** of $A \subseteq \Omega$, bel(A), quantifies the total amount of *justified specific support* given to A. It is obtained by summing all the basic belief masses given to propositions $X \subseteq A$ (and $X \neq Ø$). Let bel : $2^\Omega \to [0,1]$ where bel is called a belief function:

$$bel(A) = \sum_{Ø \neq X \subseteq A} m(X)$$

We say *justified* because we include in bel(A) *only* the basic belief masses given to subsets of A. For instance, consider two distinct elements x and y of **Ω**. The basic belief mass m({x,y}) given to {x,y} could support x if further information indicates this. However given the available information the basic belief mass can only be given to {x,y}. We say specific because m(Ø) is not be included in bel(A). Indeed Ø supports both A and $\overline{A}$, thus Ø does not support specifically A.

The unnormalized Dempster's rule of conditioning expressed with bel is:

$$bel(A|B) = bel(A \cup \overline{B}) - bel(\overline{B}) \qquad \text{for all } A \subseteq B$$

Its normalization is obtained by dividing the terms bel(A|B) by $bel(\Omega)-bel(\overline{B}) = 1-m(Ø)-bel(\overline{B})$.

1.2.5. The Pignistic Probability BetP .

In Smets (1990) and Smets and Kennes (1994), we show how to make **decisions** when the beliefs are quantified by belief functions (see also Jaffray (1988) and Stratt (1989, 1990) for other solutions based on Hurwicz index).

Before any bet can be elaborated, betting alternatives must be established. Let Ω' be the **betting frame**, i.e. the set of mutually exclusive and exhaustive options on which the bet is to be established and on which 'rewards' can be allocated freely. Let Your beliefs on a frame Ω be quantified by the normalized belief function bel. Let bel' be Your beliefs on the frame Ω' derived from bel and such that bel' and bel allocated same degrees of belief to identical subsets of Ω' and Ω (subsets that express equivalent propositions), and bel' does not express more information than bel did (i.e. bel' is the least committed belief function among those that allocate same beliefs as bel on equivalent propositions).

Let BetP be the probability function on Ω' on which bets are established,. BetP is called the pignistic probability function. It depends on Ω' and bel (through bel'). Let Γ be the 'pignistic' transformation such that:

$$\text{BetP} = \Gamma(\text{bel}, \Omega')$$

where BetP is a probability measure on Ω'.

In Smets (1990) we present the rationality requirements that lead to the unique solution for BetP = Γ(bel, Ω'):

$$\text{BetP}(\{\omega\}) = \sum_{\omega \in B \subseteq \Omega'} \frac{m'(B)}{|B|} \qquad \forall \omega \in \Omega'$$

in which case:

$$\text{BetP}(A) = \sum_{B \subseteq \Omega'} m'(B) \frac{|A \cap B|}{|B|} \qquad \forall A \subseteq \Omega' \qquad (**)$$

where |X| is the number of elements of Ω' in X and m' is the bba related to bel' (where bel' is normalized). The requirements are summarized as follows (their names are taken from Gilboa, 1989):

- linearity: $\Gamma(p\ bel_1 + (1-p)\ bel_2, \Omega') = p\ \Gamma(bel_1, \Omega') + (1-p)\ \Gamma(bel_2, \Omega')$.
- projectivity: if bel is a probability function P on Ω, then Γ(P, Ω) = P.
- efficiency: BetP(Ω') = 1.
- anonymity: BetP is not sensible to any permutation of the elements of Ω.
- false event: If an impossible element ω is added to Ω, then BetP(ω) = 0.

The solution (**) happens to be equivalent to the Shapley value (Shapley 1953), which model was also called the transferable utility games.

The major rationality requirements are that 1) decisions are based on a probability measure (and the optimization of the expected utility) and 2) the linearity. The first is justified by Savage axioms (Savage 1954). It is not satisfied by Jaffray (1988) and Stratt (1989, 1990) proposals. The second is based on the following argument. Let bel_1 and bel_2 be two belief functions on the same frame Ω in two contexts. Let the contexts be selected at random with probability p and q = 1-p, respectively. Depending on the context selected, bel_1 or bel_2 will quantify Your beliefs on Ω.

From each bel_i one can build the induced pignistic probability function $BetP_i$ on Ω (at this level the distinction between Ω and Ω' is irrelevant, so we neglect it). These $BetP_i$ are conditional probability functions on Ω, where the conditioning events have probabilities p and q. The probability function BetP before selecting the context is thus:

$$BetP = p\ BetP_1 + q\ BetP_2$$

There is another way to derive the pignistic probability before selecting the context. If bel_1 and bel_2 are the belief functions on Ω in each context, then the belief function bel on Ω before selecting the context can be shown to be:

$$bel = p\ bel_1 + q\ bel_2.$$

Knowing Your beliefs on Ω before selecting the context, You derive Your pignistic probabilities on Ω by applying the pignistic transformation to bel.

The linearity requirement corresponds to the requirement that the two derivations (combining the pignistic probability induced by each belief function or taking the pignistic probability induced by the combined beliefs) lead to the same pignistic probabilities.

Note that BetP is not a representation of Your beliefs on the betting frame Ω'. It is the additive measure induced on Ω' by Your beliefs held at the credal level (and quantified in the TBM by a belief function) when decision must be made and that must be used to compute the expected utility to be maximize in order to select the optimal decision.

1.2.6. What a Dutch Book is ?

The subjective probability P(A) given by an agent to an event A at a given time is usually defined as the *price* the agent is willing to pay to play a game against a banker where the agent receives $1 from the banker if the event A occurs and nothing if the event A does not occur. Furthermore the bet is *fair* if the agent is indifferent between being the player or the banker once the price has been fixed. Such a procedure leads to the construction of probabilities that satisfy the classical axioms of the probability measures and gives a semantic to the subjective probability measures. Unfortunately this definition does not justify why beliefs should be quantified by probability measures. In order to explain why belief should indeed be quantified by probability measures, Bayesians use the concept of Dutch Book, originally.

A Dutch Book, as introduced by de Finetti (1937), involves a set of bets on events over a domain X (a betting frame) offered by a Bookie to an agent. First the agent fixes the price of each $1 bet. The bookie decides then for each bet if the agent is the player or the banker and the amount to be bet. This set of decision is called a strategy. A Dutch Book is build by the Bookie against the agent if the Bookie can find a strategy under which the agent loses money in every case. It can be shown that the value of the bets must be based on some underlying probability measure over the possible outcomes of the betting frame X. This type of Dutch Book is qualified as Synchronic as far as no time is involved. It justified the additivity of subjective probabilities. It also justifies the Bayesian rule of conditioning provided the introduction of the possibility of bets being reimbursed if some outcome does not occur. This last form of bets fits to the concept of conditionalization as defined in Goldstein (1985), but not necessarily to the concept of a posteriori probabilities.

Suppose evolving time is involved in the betting procedure and it is agreed upon that the Bookie can buy back any pending bets after some intermediary events have occurred which outcome is known to both participants,. Then a Diachronic Dutch Book argument has been advanced to justify why beliefs should be updated by the application of the Bayesian rule of conditioning (Teller 1973, Jeffrey 1988). This argument is used to justify that the a posteriori beliefs should be derived from the initial beliefs by the application of the Bayesian rule of conditioning, just as for the conditionalization process.

We have proposed to use the transferable belief model in order to represent the quantified beliefs held by an agent. The pignistic transformation has been defined in order to build the probability measure needed to fix the prices of the bets. As far as the pignistic probability is a probability measure, no Synchronic Dutch Book can be established against the agent. The problem appears once updating is involved. Updating is performed in the transferable belief model by the application of Dempster's rule of conditioning. In general, the pignistic transformation of the updated belief function is not what would be obtained by the application of Bayes rule of conditioning applied to the initial pignistic probability function, violating thus the consistency requirement of Gilboa (1989). So it may seem that a Diachronic Dutch Book could be applied against the agent. We show that it is not the case, and therefore that the TBM is immune to the Dutch Book argument, even though beliefs are not quantified by probability measures and updating is not obtained by the application of Bayes rule.

We do not prove that the transferable belief model can resists to ANY Dutch Book. Its resistance to Synchronic Dutch Books is immediate, as bets are based on a probability measure derived from our belief function. We show also that the Diachronic Dutch Book that supposedly justifies the Bayesian updating rule cannot be used against the transferable belief model. It may be that other Dutch Books could be created, but , as far as I know, they remain to be built if they exist.

First we present an example called the ‘Peter, Paul and Mary saga’. We proceed by describing what should be the diachroninc Dutch Book applicable in that example.

Finally we show how to build the pignistic probabilities so that no diachroninc Dutch Book can be build against an agent that used the TBM to quantify his/her beliefs.

2. The Peter, Paul and Mary Scenario

2.1. The Saga.

Big Boss has decided that Mr. Jones must be murdered by one of the three people present in his waiting room and whose names are Peter, Paul and Mary. Big Boss has decided that the killer on duty will be selected by a throw of a dice: if it is an even number, the killer will be female, if it is an odd number, the killer will be male. You, the judge, know that Mr. Jones has been murdered and who was in the waiting room. You know about the dice throwing, but You do not know what the outcome was and who was actually selected. *You are also ignorant as to how Big Boss would have decided between Peter and Paul in the case of an odd number being observed.* Given the available information at time t_0, Your odds for betting on the sex of the killer would be 1 to 1 for male versus female.

At time $t_2 > t_0$[†] , You learn that if Big Boss had not selected Peter, then Peter would necessarily have gone to the police station at the time of the killing in order to have a perfect alibi. Peter indeed went to the police station, so he is not the killer. The question is how You would bet now on male versus female: should Your odds be 1 to 1 (as in the TBM) or 1 to 2 (as in the Bayesian model)

Note that the alibi evidence makes 'Peter is not the killer' and 'Peter has a perfect alibi' equivalent. The more classical evidence 'Peter has a perfect alibi' would only imply P('Peter is not the killer' | 'Peter has a perfect alibi') = 1. But P('Peter has a perfect alibi' | 'Peter is not the killer') would stay undefined and would then give rise to further discussion, which for our purpose would be useless. In this presentation, the latter probability is also 1.

2.2. The TBM Solution.

Let k be the killer. The information about the waiting room and the dice throwing pattern induces the following basic belief assignment m_0:

$k \in \Omega = \{\text{Peter, Paul, Mary}\}$

$m_0(\{\text{Mary}\}) = .5$

$m_0(\{\text{Peter, Paul}\}) = .5$

[†] The choice of index 2 will be justified later.

The .5 belief mass given to {Peter, Paul} corresponds to that part of belief that supports "Peter or Paul", could possibly support each of them, but given the lack of further information, cannot be divided more specifically between Peter and Paul.

Let $BetP_0$ be the pignistic probability obtained by applying the pignistic transformation Γ to m_0 on the betting frame which set of atoms $\mathbb{A}$ is {{Peter], {Paul}, {Mary}}.

$$BetP_0 = \Gamma(m_0, \mathbb{A})$$

with:

$$\begin{aligned} BetP_0(\{Peter\}) &= m_0(\{Peter\}) + m_0(\{Peter, Paul\}) / 2 \\ &\quad + m_0(\{Peter, Mary\}) / 2 + m_0(\{Peter, Paul, Mary\}) / 3 \\ &= .25 \end{aligned}$$

and equivalently:

$$\begin{aligned} BetP_0(\{Paul\}) &= .25 \\ BetP_0(\{Mary\}) &= .50 \end{aligned}$$

Given the information available at time t_0, the bet on the killer's sex (male versus female) is held at odds 1 to 1.

Peter's alibi induces an updating of m_0 into m_2 be Dempster's rule of conditioning:

$$m_2(\{Mary\}) = m_2(\{Paul\}) = .5$$

The basic belief mass that was given to "Peter or Paul" is transferred to Paul.

Let $BetP_2$ be the pignistic probability obtained by applying the pignistic transformation Γ to m_2 on the betting frame which set of atoms $\mathbb{B}$ is {{Paul}, {Mary}}.

$$BetP_2 = \Gamma(m_2, \mathbb{B})$$

where:

$$\begin{aligned} BetP_2(\{Paul\}) &= .50 \\ BetP_2(\{Mary\}) &= .50 \end{aligned}$$

Your odds for betting on Male versus Female would still be 1 to 1.

3. Diachronic Dutch Book.

We must distinguish between static and dynamic contexts. Updating is not considered in the first case, whereas it is in the second. Synchronic Dutch Books justify the use of a probability measure in the static case. Diachronic Dutch Books concern the dynamic case.

We do not discuss the Synchronic Dutch Book in the static TBM context as far as all synchronic bets are based on the pignistic probability BetP that is just a plain probability function. Therefore no Dutch Book can be build against the static TBM solution.

In the dynamic case, Bayesians claim they can build a Diachronic Dutch Book against the TBM solution because the pignistic probabilities derived from the updated belief functions are not those one would obtain from $BetP_0$ by the application of the Bayesian rule of conditioning on $BetP_0$ (Teller 1973, Jeffrey 1988). We show that the argument is not against the TBM and its updating rule, but that the construction of the pignistic probabilities must take in due account the information underlying the construction of the Diachronic Dutch Book, i.e. that an experiment is going to be run before bets will be settled. We illustrate our case with the Peter, Paul and Mary saga.

Using the figures obtained with the TBM solution, Jeffrey (1988) proposes the following strategy in order to build a Dutch Book.

	You win if killer is				Bookie's	fair
	Peter	Paul	Mary	P_w	price	price
B1	0	0	1	.50	24	12
B2	1	0	0	.25	16	4
B3	0	1	1	.75	4	3
$BetP_0$	.25	.25	.50			

Table 3.1. Data about the three bets B1, B2 and B3.

Let the three \$1 bets B1, B2 and B3 given in table 3.1. At time t_0, Your pignistic probabilities $BetP_0$ for the three options are those given at the bottom of the table. The probability of winning for each bet is given in column P_w. The Bookie sells You the three bets at \$24, \$16 and \$4 respectively. Given Your pignistic probabilities $BetP_0$, the 'fair' price You accept to pay for the three bets are given in the rightmost column. For instance, You pay \$12 to buy the ticket B1, You will receive \$24 if Mary is the killer, and nothing otherwise. The total cost of the three bets is \$19 at time t_0.

Suppose an experiment is going to be run at time t_1, the outcome of which (known by both You and the Bookie) will be either 'Peter is the killer' or 'Peter is not the killer'. If you learn that Peter is the killer, the bets are settled and You loose \$3. If you learn that Peter is not the killer, the Bookie's trick is that he buys back the B1 ticket at the new value that You would give to it at time t_1. In the TBM, after learning 'not Peter' bel_0 is updated into $bel_1 = bel_0(.|\text{not Peter})$ by conditioning bel_0 on 'not Peter'. One obtains $m_1(\text{Paul}) = .5$ and $m_1(\text{Mary}) = .5$. Then the 'fair' price You give to B1 is based on $BetP_1 = \Gamma(bel_1, \{\{\text{Paul}\}, \{\text{Mary}\}\})$. One obtains $BetP_1(\text{Paul}) = BetP_1(\text{Mary}) = .5$. Therefore B2 price is still \$12 = \$0*$BetP_1$(Paul) + \$24*$BetP_1$(Mary).

If You learn	B1	B2	B3	Net Gain
Peter is the killer	0	16	0	16 - 19 = -3
Peter is not the killer	0	0	4	4 + 12 - 19 = -3

Table 3.2. Gain received from the three bets after learning that Peter is the killer or isn't it. (in which case the Bookie buys back bet B1 at $12).

The net outcome of these bets are given in table 3.2. Net gain results from the price paid by the settled bets ($16 or $4, respectively), the $19 paid at t_0 (and the $12 the Bookie might pay to You at t_1 if you learn that Peter is not the killer). In both cases, You loose $3, and a Dutch Book has been build against You. This argument is the one used to justify the use of the Bayesian rule of conditioning.

But the argument is not correctly applied as we show now. The information about the experimental set up must be introduced in the TBM solution. The solution presented in section 1.2.5. deals with the case where all the available information is summarized in bel. In the present context, You know more, You know about the experiment that will be run. The Dutch Book was obtained because the pignistic probability was erroneously computed.

The pignistic probability $BetP_0$ of table 3.1 are those You would use if You had not known about the Peter experiment. They were perfectly valid as You might have win (if Mary turned out to be the killer). The Bookie build a Dutch Book only because he knows that the Peter experiment was going to be run. Should You have known about the experiment, You would not have used the pignistic probabilities $BetP_0$ of table 3.1, but another set of pignistic probabilities that reflect the information about the to be run experiment. In sections 4 and 5, we develop in detail how to build the pignistic probabilities on Ω when You know that an experiment will be run which outcome is relevant to Your beliefs on Ω. We show that such pignistic probabilities are immune to Dutch Books.

4. Outcome of the Peter Experiment at time t_1 .

Let us rephrase the scenario. At time t_0, You and the Bookie know the same information about the Peter, Paul and Mary saga as previously. But you both know also that at time $t_1 > t_0$ a 'Peter sensor' will tell to both of you if Peter is or is not the killer. If Peter is not the killer, the Bookie is free to propose new bets or to buy back the pending one (B1). At time $t_2 > t_1$, the bets will be settled if Peter happens not to be the killer (they were already settled at t_1 if Peter is the killer).

You are thus facing two sets of bets: those at t_0 and those at t_1 if the Peter sensor says 'not Peter' (see table 4.1). If You learn 'not Peter', You will update Your initial belief function bel_0 into $bel_1 = bel_0(.|\text{not Peter})$ by Dempster's rule of conditioning. Then $m_1(\{Paul\}) = m_1(\{Mary\}) = .5$. Therefore Your pignistic probabilities are $BetP_1 = \Gamma(bel_1, \{\{Paul\}, \{Mary\}\})$, i.e., $BetP_1(\{Paul\}) = BetP_1(\{Mary\}) = .5$. These are the probabilities You will use at time t_1 if you learn ‘not Peter’. At time t_0, You build Your pignistic probability $BetP_X$ that You will observe ‘Peter’ or ‘not Peter’ at t_1. So $BetP_X = \Gamma(bel_0, \{\{Peter\}, \{Paul, Mary\}\})$. Hence $BetP_X(\{Peter\}) = .25$ and $BetP_X(\{Paul, Mary\}) = .75$. The pignistic probabilities $BetP_\Omega$ to be used at time t_0 to compute the fair price of the proposed bets are then (see table 4.1):

$BetP_\Omega(\{Peter\}) = .25$

$BetP_\Omega(\{Paul\}) = .375$

$BetP_\Omega(\{Mary\}) = .375$

and not the .25, .25, .50 as given in the solution where it was not known about the Peter experiment to be run. As far as all bets, those at t_0 and those at t_1, are linked through an underlying join probability function, and those at t_1 can be derived from those at t_0 through the application of the Bayesian rule of conditioning, no Diachronic Dutch Book can be build against You, as shown in Teller’s work.

You learn X at t_1	The killer ϖ is Peter	Paul	Mary	$BetP_X$ on X / -X
X = Peter is the killer	1.	0.	0.	.25
-X = Peter is not the killer	0.	.5	.5	.75
$BetP_\Omega$ on ϖ at t_0	.25	.375	.375	

Table 4.1. Values of the conditional probabilities You would use at t_1 if You learn X or -X. Prior probabilities $BetP_X$ that You will learn X or -X. Probabilities $BetP_\Omega$ used to bet on the killer at t_0 knowing about the X experiment that will be run.

Therefore we have shown how the TBM resists to the Dutch Book argument. The TBM does not update beliefs through the Bayesian rule of conditioning, but resists nevertheless against Dutch Booking by considering appropriately the experimental set up involved. Knowing that a ‘Peter sensor’ is going to be used is a piece of information that must be taken in consideration. Once that is done, You are safe against any Dutch Book as all Your bets are executed by using pignistic probabilities that are consistent with an underlying join probability function on the product space Ωx{possible outcomes of the experiment}.

$BetP_0$ given in table 3.1 corresponds to the pignistic probabilities derived from the knowledge available at t_0 whereas $BetP_0$ in table 4.1 corresponds to the pignistic probabilities derived from the same information plus the knowledge about the ‘Peter sensor’ experiment to be run at t_1 and the ability for the Bookie to buy back any unsettled bet at t_1 . (If the Bookie could not buy back pending bets at t_1, the appropriate pignistic probabilities are of course the $BetP_0$ of table 3.1).

The originality achieved with the TBM resides in the fact that the probabilities used at t_0 will not be the same if You ignore the Peter experiment or if You know about it (the BetP of tables 3.1 or 4.1). One could be tempted to create a Dutch Book by mixing bets in the two contexts, ignoring or knowing about the Peter experiment. This is not achievable as one can not accept a context where You simultaneously ignore and know the same thing: either You don't know or You know something. Such Dutch Books could only be achieved in absurd worlds.

5. General Solution.

5.1. Observation is: '$\varpi \in \mathbf{B} \subseteq \Omega$'

Let $m_0:2^\Omega \rightarrow [0, 1]$ be Your basic belief assignment at time t_0 and bel_0 its related belief function, where $bel_0(A)$, $\forall A \subseteq \Omega$, quantifies Your degree of belief that the actual state of affair ϖ belongs to A. Let it be known that at time t_1 an experiment will be run which outcome will be an atom of a subalgebra $\mathcal{B}$ of 2^Ω. Let $\mathbb{B} = \{B_1, B_2, ..., B_n\}$ be the set of the atoms of $\mathcal{B}$. Each B_i is itself a non empty set of elements of Ω: $B_i = \{\omega_{i1}, \omega_{i2}, ...\omega_{in_i}\} \subseteq \Omega$.

Let m_i be the basic belief assignment obtained by conditioning m_0 on B_i by Dempster's rule of conditioning, with bel_i its related belief function. Let $BetP_i = \Gamma(bel_i, B_i)$. $BetP_i(\{\omega_{ij}\})$ is the pignistic probability of $\omega_{ij} \in B_i$ derived from bel_i on the elements of Ω in B_i, i.e. the betting frame corresponds to the elements of B_i. Let $BetP_0 = \Gamma(bel_0, \mathbb{B})$. $BetP_0(B_i)$ is the pignistic probability of $B_i \in \mathbb{B}$ derived from bel_0 on the atoms of $\mathcal{B}$, i.e. the betting frame corresponds to the atoms of $\mathcal{B}$.

Then the pignistic probability BetP* at t_0 on the elements of Ω is:

$$\forall B_i \in \mathbb{B} \quad \forall \omega \in B_i, \qquad BetP^*(\omega) = BetP_i(\omega)\, BetP_0(B_i)$$

$$\text{and} \quad \forall A \subseteq \Omega \qquad BetP^*(A) = \sum_{\omega \in A} BetP^*(\omega)$$

This formula is derived from the fact that $BetP_i(\omega)$ corresponds to the conditional probability of ω given B_i and $BetP_0(B_i)$ is the marginal probability of B_i.

5.2. General case.

Let $m_0:2^\Omega \rightarrow [0, 1]$ be Your basic belief assignment at time t_0 and bel_0 its related belief function where $bel_0(A)$, $\forall A \subseteq \Omega$, quantifies Your degree of belief that the actual state of affair ϖ belongs to A. Let it be known that at time t_1 an experiment will be run which outcome will be an element of a frame X = {x_1, x_2...x_n}. How to assess Your pignistic probabilities $BetP_0$ on Ω given m_0 and the knowledge about the X experiment?

Let $m_\omega: 2^X \rightarrow [0, 1]$ be Your basic belief assignment at time t_0 about the outcome of the experiment if $\varpi = \omega$, $\forall \omega \in \Omega$, and bel_ω its related belief function. Consider the space ΩxX. For each $\omega \in \Omega$, bel_ω is the conditional belief functions induced on X under the condition that $\varpi = \omega$. It translates Your beliefs about which experimental outcome will occur if $\varpi = \omega$. Beside bel_0 is Your a priori belief on Ω. We have shown in Smets (1978, 1993) how to build the underlying belief function $bel_{\Omega xX}$ on ΩxX that integrates the information contained in bel_0 and all the bel_ω. Let $bel_\Omega(.|x)$ be the belief function on Ω when conditioning $bel_{\Omega xX}$ on $x \in X$. The full relations are given in Smets (1993).

For each $x \in X$, build $BetP_x = \Gamma(bel_\Omega(.|x), \Omega)$, the pignistic probability induced by $bel_{\Omega xX}$ on Ω when x is the case. This would be the probabilities You would use on Ω if the experimental outcome is $x \in X$.

Build then $BetP_X = \Gamma(bel_{\Omega xX}, X)$, the pignistic probability derived from the joint belief function $bel_{\Omega xX}$ on the betting frame X.

Then the pignistic probability BetP* at t_0 on the elements of Ω is:

$$\forall A \subseteq \Omega, \qquad BetP^*(A) = \sum_{x \in X} BetP_x(A)\, BetP_X(x)$$

This relation corresponds to the classical relation:

$$\forall A \subseteq \Omega \qquad P(A) = \sum_{C_i \in \mathbb{C}} P(A|C_i)\, P(C_i)$$

where the C_i are the elements of a partition $\mathbb{C}$ of Ω.

The probabilities BetP*(ω) are those You use at time t_0 when You know there will be an experiment which outcome at t_1 will affect Your beliefs over Ω and new bets on Ω could be proposed after knowing the outcome of the experiment. Let BetP' = $\Gamma(bel_0, \Omega)$ be the probabilities you would use if there was no known experiment going to be run before deciding of the final outcome of the bet on the frame Ω. The difference between BetP* and BetP' reflects the impact of the knowledge of the to-be-run experiment and the ability for the Bookie to buy back some pending bets.

5.3. Example: Outcome is '¬Peter' or '¬Mary'

As an example of the general case, suppose that the experimental outcome could be '*¬Peter*' or '*¬Mary*' (italics characterize the experimental outcomes). Let the experiment be called the *X* experiment. The outcome is '¬*Peter*' if the killer is Mary, '¬*Mary*' if the killer is Peter, but what if the killer is Paul? Suppose You believe that both answer are equiprobable when Paul is the killer. What would be BetP* in that case?

The initial frame is Ωx*X* = {Peter, Paul, Mary}x{¬*Peter*, ¬*Mary*}. Tables 5.1 and 5.2 illustrate the construction of the various belief functions and pignistic probability functions. The basic belief assignment m^c on Ωx*X* that results from the conditional belief functions on *X* given $\omega\in\Omega$ is[1]:

m^c({(Peter,¬*Mary*), (Paul, ¬*Peter*), (Mary, ¬*Peter*)}) = .5

m^c({(Peter,¬*Mary*), (Paul, ¬*Mary*), (Mary, ¬*Peter*)}) = .5

	m_ω			m^c			coin data			$m_{\Omega xX}$			m_X
	Pet	Pau	Mar	Pet	Pau	Mar	Pet	Pau	Mar	Pet	Pau	Mar	
¬*Pet*	.0	.5	1.		.	.					.	.5	.50
				.5	.5			.5	.5	.25			.25
¬*Mar*	1.	.5	.0	.	.					. .25 .			.25

Table 5.1. Values of m_ω where Pet, Pau and Mar denote Peter, Paul and Mary, respectively. Construction of m^c, and the belief induced by the coin data. Construction of $m_{\Omega xX}$ by combining the last two belief functions by Dempster's rule of combination. Construction of m_X. Lines and squares designate the subsets of Ωx*X* to which the bbm are given.

	m_x			$BetP_x$			$BetP_X$
	Pet	Pau	Mar	Pet	Pau	Mar	
¬*Pet*	.0	.25	50	.0	1/3	2/3	.625
	.25						
¬*Mar*	.25	.0	.0	3/4	1/4	.0	.375

Table 5.2. Construction of m_x, $BetP_x$ and $BetP_X$ where $x\in X$ = {¬*Peter* ,¬*Mary*}.

The basic belief assignment $m_{\Omega xX}$ on Ωx*X* that combines m^c with the prior information on Ω (the coin experiment: .5 on male and .5 on female) is:

$m_{\Omega xX}$({(Peter,¬*Mary*), (Paul, ¬*Peter*)}) = .25

[1] This is the only belief function such that its conditioning on the elements of Ω restitute the original conditional belief functions on X. The general solution is presented in Smets (1993)

$m_{\Omega x X}$({(Peter,¬*Mary*), (Paul, ¬*Mary*)}) = .25
$m_{\Omega x X}$({(Mary, ¬*Peter*)}) = .5
$m_{\Omega x X}$ integrates all prior knowledge relative to the bets.

Let m_X be the bbm induced by $m_{\Omega x X}$ on *X*:
m_X (¬*Peter*) = .5
m_X (¬*Mary*) = .25
m_X (¬*Peter* or ¬*Mary*) = .25
m_X quantifies Your beliefs induced by $m_{\Omega x X}$ on *X*.

Let $m_{\neg Peter}$ and $m_{\neg Mary}$ be the bbm induced on Ω after conditioning $m_{\Omega x X}$ on ¬*Peter* or ¬*Mary*, respectively. They $m_{\neg Peter}$ and $m_{\neg Mary}$ quantifies Your beliefs induced by $m_{\Omega x X}$ on Ω after knowing that the experimental outcome is ¬*Peter* or ¬*Mary*, respectively. Their values are:
$m_{\neg Peter}$(Paul) = .25 and $m_{\neg Peter}$(Mary) = .50
$m_{\neg Mary}$(Peter or Paul) = .25 and $m_{\neg Mary}$(Peter) = .25
These belief functions will be normalized (by .75 and .50, respectively) before applying the pignistic transformation.

Then:
$BetP_X$ = Γ(bel_X , {{¬*Peter*} , {¬*Mary*}} is used to bet on the experimental outcome.
$BetP_{\neg Peter}$ = Γ($bel_{\neg Peter}$, {{Paul} , {Mary}}) is used to bet on Paul versus Mary if the experimental outcome is ¬*Peter*.
$BetP_{\neg Mary}$ = Γ($bel_{\neg Mary}$, {{Paul} , {Peter}}) is used to bet on Paul versus Peter if the experimental outcome is ¬*Mary*.

For instance: $BetP_X$(¬Peter) = .25/2 + .5 = .625
and $BetP_{\neg Peter}$(Paul) = .25/.75.

Table 5.3 presents the various probabilities. Your bets at t_0 on Ω are based on BetP* (and not the .25, .25, .50 You should have used if You had not be aware of the experimental set up).

$BetP_x$	Peter	Paul	Mary	$BetP_X$
¬*Peter*	0.	.25/.75	.50/.75	.625
¬*Mary*	.375/.50	.125/.50	0.	.375
BetP*	.281	.302	.417	

Table 5.3. Values of the conditional pignistic probabilities $BetP_x$ on Ω where $x \in X$ = {¬*Peter* ,¬*Mary*}, of the marginal pignistic probabilities $BetP_X$ on *X* and of the pignistic probabilities BetP* on Ω used at t_0 (before observing the outcome of the *X* experiment).(.302 = (.25/.75).625 + (.125/.50).375, etc...)

Coming back to the B1, B2 and B3 bets considered in section 3, the fair prices for the three bets at t_0 (based on BetP*, see table 5.2) are: (B1, $10) , (B2, $4.5), (B3, $2.875).

5.4. vacuous prior on *X* when Paul is the killer.

We consider the same example as in section 5.3 but where we are in a state of total ignorance on how the *X* outcome will be selected if the killer is Paul, i.e. the belief function over *X* given Paul is represented by a vacuous belief function.

	m_ω			m^c			coin data			$m_{\Omega x X}$			m_X	
	Pet	Pau	Mar	Pet	Pau	Mar	Pet	Pau	Mar	Pet	Pau	Mar		
¬*Pet*	.0	.	1.		.	.					.	.5	.5	
		1.			1.		.5		.5		.5			.5
¬*Mar*	1.	.	.0	.	.					.	.		.0	

Table 5.4. Same information as table 5.1, except for m $_{\omega\text{=Paul}}$.

	m_X			$BetP_X$			$BetP_X$
	Pet	Pau	Mar	Pet	Pau	Mar	
¬*Pet*	.0	.5	5	.0	.5	.5	.75
	.5						
¬*Mar*	.0	.0	.0	.5	.5	.0	.25

Table 5.5. Same information as table 5.2, except $m_{\omega=\text{Paul}}$ represents total ignorance.

The initial frame is ΩxX = {Peter, Paul, Mary}x{¬*Peter*, ¬*Mary*}. Tables 5.4 and 5.5 illustrate the construction of the various belief functions and pignistic probability functions. The basic belief assignment m^c on ΩxX that results from the conditional belief functions is:

m^c({(Peter,¬*Mary*), (Paul, ¬*Peter*), (Paul, ¬*Mary*) (Mary, ¬*Peter*)}) = 1

The basic belief assignment $m_{\Omega xX}$ on ΩxX that combines m^c with the prior information on Ω is:

$m_{\Omega xX}$({(Peter,¬*Mary*), (Paul, ¬*Peter*), (Paul,¬*Mary*)}) = .50

$m_{\Omega xX}$({(Mary, ¬*Peter*)}) = .5

Table 5.6 presents the various probabilities computed as for those in table 5.1.

$BetP_x$	Peter	Paul	Mary	$BetP_X$
¬*Peter*	0.	.50	.50	.75
¬*Mary*	.25/.50	.25/.50	0.	.25
BetP*	.125	.50	.375	

Table 5.6. Values of the conditional pignistic probabilities $BetP_x$ on Ω where $x \in X = \{\neg Peter, \neg Mary\}$, of the marginal pignistic probabilities $BetP_X$ on X and of the pignistic probabilities BetP* on Ω used at t_0 (before observing the outcome of the X experiment).

Coming back to the B1, B2 and B3 bets considered in section 3, the fair prices for the three bets at t_0 (based on BetP*, see table 5.5) are: (B1, \$9) , (B2, \$2), (B3, \$3.5).

6. The TBM solutions are not compatible with some Bayesian solution.

To enhance the difference between the Bayesian and the TBM approaches, we present a simplified example that shows that the TBM solutions obtained when ignoring that a given experiment will be run and those derived when knowing about the experiment cannot be achieved simultaneously by some underlying Bayesian analysis.

Ω	m_0	$m_{\{a,b\}}$	$m_{\{c\}}$
{a}	.0	.5	
{b}	.0		
{c}	.0		1.
{a,b}	.0	.5	
{a,c}	.5		
{b,c}	.0		
{a,b,c}	.5		

Table 6.1. Basic belief assignments m_0, $m_{\{a,b\}}$ and $m_{\{c\}}$ one would build on Ω before or after learning the various outcomes of test E.

Suppose a frame Ω = {a, b, c} and let ϖ be the actual value, unknown to You. Let m_0 quantifies Your beliefs that ϖ belongs to the various subsets of Ω (table 6.1). Let E be an experiment that tests if $\varpi \in \{a,b\}$ or not. Table 6.1. presents m_0 and the basic belief assignment $m_{\{a,b\}}$ and $m_{\{c\}}$ one would build on Ω after learning the outcome of the test E.

$BetP_x$	a	b	c
x = {a,b}	.75	.25	.00
x = {c}	.00	.00	1.0
$BetP_0$	5/12	2/12	5/12

Table 6.2. The pignistic probabilities $BetP_x$ on Ω for each outcome x of the E experiment, and $BetP_0$, those derived if one ignores that an experiment will be run.

Table 6.2 presents the various pignistic probabilities one would use to bet if one knew the outcome of the experiment and if one ignored that an experiment will be run. There is no $\alpha \in [0, 1]$ such that $BetP_0 = \alpha\, BetP_{\{a,b\}} + (1-\alpha)\, BetP_{\{c\}}$. So BetP cannot be consider as the result of the conditionalisation by Bayes rules of a probability measure on Ωx{{a,b}, {c}} such that its marginal is $BetP_0$.

7. Conclusion.

We have shown how the Peter, Paul and Mary saga resists to Dutch Books, even though updating is performed according to the transferable belief model by Dempster's rule of conditioning and not by the classical Bayesian rule of conditioning. The Dutch Book argument does not provide a proof that updating has to be performed according to the Bayesian rule. Of course we do not require that conditioning and a posteriori probabilities should be the same. We see a difference between a bet where some options would lead to a cancellation of the bet with reimbursement of the paid price (that is to be based on conditionalized probabilities) and a bet after learning that some conditioning event has occurred (that is to be based on a posteriori probabilities).

Consider a space Ω and two sets A, B. Let a bet on A∩B versus -A∩B where the ticket price is paid back if -B. Such a bet would be obtained by conditioning the pignistic probabilities on Ω by Bayes rule. It would avoid a Synchronic Dutch Book. Temporal Coherence Principle requires that such a bet would be the same as the one on A versus -A once the actual world ϖ is known to belong to B. In the TBM, this requirement is not assumed. The second bet is based on different pignistic probabilities derived from the belief obtained by conditioning the initial belief on B by Dempster's rule of conditioning, what leads in general to a different bet on Ω.

Acknowledgments.
The author is indebted to P. Garbolino and J. Y. Jaffray for many suggestions about this work.

Bibliography.

DE FINETTI B. (1937) La prévision: ses lois logiques, ses sources subjectives. Ann. Inst. H. Poincarré, 7:1-68.

DEGROOT M.H. (1970) Optimal statistical decisions. McGraw-Hill, New York.

GILBOA I. (1989) Additivizations of non additive measures. Meth. Operat. Res. 14:1-17.

GOLDSTEIN M. (1985) Temporal coherence. in Bayesian Statistics 2, Bernardo J.M., Degroot M.H., Lindley D.V. and Smith A.F.M. eds. North Holland, Amsterdam, 231-248.

JAFFRAY J.Y. (1988) Application of linear utility theory for belief functions. in BOUCHON B., SAITTA L. and YAGER R.R. eds. Uncertainty and Intelligent Systems, Springer-Verlag, Berlin. p. 1-8

JEFFREY R. (1988) Conditioning, kinematics, and exchangeability. in Causation, Chance, and Credence, Skyrms B. and Harper W.L. eds., Reidel, Dordrecht,,vol.1, 221-255.

KLAWONN F. and SMETS Ph. (1992) The dynammic of belief in the transferable belief model and specialization-generalization matrices. in Dubois D., Wellman M.P., d’Ambrosio B. and Smets P. Uncertainty in AI 92. Morgan Kaufmann, San Mateo, Ca, USA, 1992, pg.130-137.

KRUSE R., NAUCK D. and KLAWONN F. (1991) Reasoning with Mass Distributions. In: B.D. D'Ambrosio, P. Smets, P.P. Bonisonne (eds.): Uncertainty in Artifical Intelligence, Morgan Kaufmann, San Mateo, pp.182-187.

NGUYEN T. H. and SMETS Ph. (1993) On Dynamics of Cautious Belief and Conditional Objects. Int. J. Approx. Reas. 8:89-104.

SAVAGE L.J. (1954) Foundations of Statistics. Wiley, New York.

SHAFER G. (1976) A mathematical theory of evidence. Princeton Univ. Press. Princeton, NJ.

SHAPLEY L.S. (1953) A value for n-person games. In Contributions to ther Theory of Games, vol. 2, eds. H. Kuhn and A.W. Tucker. Princeton University Press, pp. 307-317.

SMETS P. (1978) Un modèle mathématico-statistique simulant le processus du diagnostic médical. Doctoral dissertation, Université Libre de Bruxelles, Bruxelles, (Available through University Microfilm International, 30-32 Mortimer Street, London W1N 7RA, thesis 80-70,003)

SMETS P. (1988) Belief functions. in SMETS Ph, MAMDANI A., DUBOIS D. and PRADE H. eds. Non standard logics for automated reasoning. Academic Press, London p 253-286.

SMETS P. (1990) Constructing the pignistic probability function in a context of uncertainty. Uncertainty in Artificial Intelligence 5, Henrion M., Shachter R.D., Kanal L.N. and Lemmer J.F. eds, North Holland, Amsterdam , 29-40.

SMETS P. (1992) The nature of the unnormalized beliefs encountered in the transferable belief model. in Dubois D., Wellman M.P., d’Ambrosio B. and Smets P. Uncertainty in AI 92. Morgan Kaufmann, San Mateo, Ca, USA, 1992, pg.292-297.

SMETS P. (1993a) Belief functions: the disjunctive rule of combination and the generalized Bayesian theorem. Int. J. Approximate Reasoning.

SMETS P. (1993b) Jeffrey’s rule of conditioning generalized to belief functions. Uncertainty in AI 93.

SMETS P. (1993c) An axiomatic justifiaction for the use of belief function to quantify beliefs. IJCAI'93 (Inter. Joint Conf. on AI), Chambery, September 1993.

SMETS P. and KENNES R. (1994) The transferable belief model. To appear in Artificial Intelligence.

SMITH C.A.B. (1961) Consistency in statistical inference and decision. J. Roy. Statist. Soc. B23:1-37.

SMITH P. and JONES O.R. (1986) The philosophy of mind, an introduction. Cambridge University Press, Cambridge.

STRAT T.M. (1989) Making decisions with belief functions. Proc. 5th Workshop on Uncertainty in IA. Windsor, 351-360. XX

STRAT T.M. (1990) Decision analysis using belief functions. Int. J. Approx. Reasoning 4: 391-418

TELLER P. (1973) Conditionalization and Observation. Synthesis 26:218-258.

If You learn	Outcome	Bookie buys	Net Gain	BetP
	Peter	nothing	-1.4	.281
	Paul	nothing	-13.4	.302
	Mary	nothing	10.6	.417
	Expected		0.0	
¬*Peter*		B1 at $16	2.6	
¬*Peter*	Paul	nothing	-13.4	.33
	Mary	nothing	10.6	.66
	Expected		2.6	
¬*Mary*	Peter	nothing	-1.4	.75
	Paul	nothing	-13.4	.25
			-4.4	
¬*Mary*	Peter	B2 at $12	-5.4	.75
	Paul	B2 at $12	-1.4	.25
	Expected		-4.4	
¬*Mary*	Peter	B3 at $1	-.4	.75
	Paul	B3 at $1	-16.4	.25
	Expected		-4.4	
¬*Mary*	Expected	B2+B3 at $13	-4.4	

Table 5.4. Same net gains as in table 5.3 the relevant pignistic probabilities and the expected gain of each strategies before or after observing the outcome of the *X* experiment.

If You learn	Outcome	Bookie buys	Net Gain	BetP
	Peter		1.5	.125
	Paul		-10.5	.500
	Mary		13.5	.375
	Expected		0.0	
¬*Peter*		B1 at $16	1.5	
	Paul		-10.5	.50
	Mary		13.5	.50
	Expected		1.5	
¬*Mary*	Peter	B2 at $12	-6.5	.50
	Paul	B2 at $12	-2.5	.50
	Expected		-4.5	
¬*Mary*	Peter	B3 at $1	3.5	.50
	Paul	B3 at $1	-12.5	.50
	Expected		-4.5	
¬*Mary*		B2+B3 at $13	-4.5	

Table 5.8. Same net gains as in table 5.7, the relevant pignistic probabilities and the expected gain of each strategies before or after observing the outcome of the *X* experiment.

Let u_ω be the gain (the amount of money or its utility if utility of money is not linear) You would receive if $\varpi=\omega$. The value of a bet at time t_0 is the expected value $E_0(u_\omega)$ of the gain u_ω where the expectation is taken with the probabilities BetP*(ω). The value of these bets at time t_1 is the expected value $E_x(u_\omega)$ of the gain where the expectation is taken with the probabilities $BetP_x(\omega)$. By construction one has always:

$$\min_{x\in X} E_x(u_\omega) \leq E_0(u_\omega) \leq \max_{x\in X} E_x(u_\omega)$$

as it should be.

?????3.5. The Rules of Combination.

Suppose two belief functions bel_1 and bel_2 induced by two 'distinct' pieces of evidence. The question is to define a belief function $bel_{12}=bel_1\oplus bel_2$ resulting from the **combination** of the two belief functions, where the ⊕ symbolizes the combination operator. Shafer proposed to use Dempster's rule of combination in order to derive bel_{12}. The underlying intuitive idea is that the product of two bbm $m_1(X)$ and $m_2(Y)$ induced by the two distinct pieces of evidence on Ω supports X∩Y, hence:

$$m_{12}(A) = \sum_{X\cap Y=A} m_1(X)\cdot m_2(Y)$$

Dubois and Prade (1986a), Smets (1990a), Klawonn and Schwecke (1990), Klawonn and Smets (1992) and Hajek (1991) provide different justifications for the origin and the unicity of this rule. These justifications are obtained without introducing some underlying probability concepts. They are based essentially on the associativity and commutativity properties of the combination operator.

Dempster's rule of combination is a rule to combine conjunctive pieces of information. Let bel_1 and bel_2 be the belief functions induced by the two distinct pieces of evidence E_1 and E_2, respectively. Then bel_{12} is the belief function induced on Ω by the conjunction 'E_1 and E_2'. In Smets (1993a) we present a combination rule that permits the derivation of the belief function induced on Ω by the disjunction of E_1 and E_2. It corresponds to a situation where you could assess Your belief on Ω if E_1 were true, Your belief on Ω if E_2 were true, but You only know that the disjunction 'E_1 or E_2' is true.

In a Bayesian analysis, $BetP^*_1$ and $BretP^*_2$ are the conditional probability functions on Ω. Therefore Your belief on Ω depends on the knowledge of which experiment will be run, what seems erroneous. Your initial belief on Ω reflects Your opinion about which element of Ω corresponds to ϖ. It should be the same whatever experiment is going to be conducted. Therefore the TBM solution is not a particular case of some Bayesian analysis.

Lert us first consider the case where both You and the Bookie do not know of any experiment that is going to be run. Both of you only knows about information available at t_0. What information could you learn before t_2 (at t_1)? You could learn, ‘Peter’, ‘Paul’, ‘Mary’, ‘¬Peter’, ‘¬Paul’, ‘¬Mary’. We examine the outcome in the six cases.

Case	Learned	B1	B2	B3	Net Gain	Bookie buys back:
C1	Peter	0	16	0	16 -19 = -3	
C2	Paul	0	0	4	4 - 19 = -15	
C3	Mary	24	0	4	28 - 19 = +9	
C4	¬Peter	0	0	4	4 + 12 - 19 = -3	B1 at $12.
C5	¬Paul	0	0	0	?	B1, B2 and/or B3
C6	¬Mary	0	0	0	- 19 + 10 = -9	B2 at $8 and B3 at $2

Table 3.3. Gains received in the three bets after learning ‘Peter’, ... ‘¬Mary’. Net gains in each case, and after letting the bookie buy back some bets according to the optimal (for him) strategy.

The case C1, C2 and C3 are trivial.

In case C4, the B1 ticket is still pending after learning ¬Peter. Its fair price given $BetP_1$ is $12. The Bookie buys it back, and you face a loss. In fact after learning ¬Peter, You were in the following situation. You paid $19, receive $4 from B3 and You would receive 24 if Mary is the killer. So Your net gains are: +$9 if Mary is the killer, -$15 is Paul is the killer. Your future is to win 9 or loose 15. Maybe would You feel that selling back Your B1 ticket at $12 will reduce the risk of loosing $15. But this type of argument is not related to the Dutch Book bussiness. Duth Booking means building a set of bets that lead to a sure loss. Up to here, no such sets has been build as You will win if You learn Mary (case C3).

Case C5 is more messy. Indeed none of the three bets is settled. Their fair prices are updated with $BetP_2$(Peter) = $BetP_2$(Mary) = .5 into: (B1, $12), (B2, $8), (B3, $2). If You keep al three tickets, Your future net gain is: +$9 if Mary is the killer, -$3 if Peter is the killer. Table 3.4 presents Your net gains according to the Bookie possible strategies. None leads to a sure loss for You, so the bookie cannot trick you in case C5.

Bookie buys back	Bookies pays	Net Gainf if killer is Peter	Mary
none	0	16 - 19 = -3	24 + 4 - 19 = 9
B1	12	16 - 19 + 12 = 9	4 -19 + 12 = -3
B2	8	- 19 + 8 = -11	24 + 4 - 19 + 8 = 17
B3	2	16 - 19 + 2 = -1	24 - 19 + 2 = 7
B1, B2	20	- 19 + 20 = 1	4 - 19 + 20 = 5
B1, B3	14	16 - 19 + 14 = 11	- 19 + 14 = -5
B2, B3	10	- 19 + 10 = - 9	24 - 19 + 10 = 15
B1, B2, B3	22	- 19 + 22 = 3	- 19 + 22 = 3

Table 3.4. Net gains of each outcomes according to the Bookie strategy for buying back some open bets after knowing ¬Peter.

In case C6, Your future is bleak. You loose $3 if Peter is the killer, and $15 if Paul is the killer. The bookie can buy the two tickets for $10 (their value according to $BetP_2$(Peter) = $BetP_2$(Paul) = .5, in which case You have a sure loss of $9.

In conclusion, there are winning cases in some of the six cases: for sure in C3, and maybe in C5. Therefore the Bookie cannot build a Dutch Book against You.

Suppose the set up is such that an experiment is going to be run (the C4, C5 and C6 cases), from which both of you might learn that one suspect is not the killer but which suspect might be eliminated is unknown at t_0. The outcome of the experiment is such that we will end up in case C4, C5 or C6. As You can win in C5, there is no Dutch Book against You.

Suppose the experimental set up corresponds to a 'Peter sensor', i.e., it can only tell 'Peter' or '¬Peter' at time t_1. (Symmetrically, one could consider a 'Mary semsor' or a 'Paul sensor'). If You and the Bookie knew about the fact that at t_1, you would know about Peter being the killer or not, then the Bookie could build a Dutch Book if the data used in case C4 were the appropriate one. But they are not. The 'Peter sensor' information has not been appropriately handled.

Table 4.1 presents the fair prices for the three bets. Their total cost is $16. With such prices, the Bookies can not build a Dutch Book. The fair price fro B1 at t_1 afeter learning '¬Peter' is $12 as $BetP_1(\{Paul\}) = BetP_1(\{Mary\}) = .50$.

Before 'Peter Sensor' outcome.				
	You win if killer is			Fair
	Peter	Paul	Mary	price
B1	0	0	24	9
B2	16	0	0	4
B3	0	4	4	3
$BetP_0$	.25	.375	.375	

Table 4.1. Outcome of the three bets, pignistic probabilities of the three outcomes, and values of the three bets.

The next table 4.2 presents Your net gains in the various possible contexts.

If You learn	B1	B2	B3	Net Gain	Bookie buys
Peter is the killer	0	16	0	16 - 16 = 0	
Peter is not the killer		0	4	4 + 12 - 16 = 0	B1 at $12
Paul is the killer	0	0	4	4 - 16 = -12	
Mary is the killer	24	0	4	28 - 16 = 12	

Table 4.2. Net gains after observing the outcome of the 'Peter sensor'. Lines 3 and 4, net gains if the Bookie does not buy back the B1 bet .

Table 5.2 presents also the fair prices of the three bets after observing ¬*Peter* or ¬*Mary*.

	You win if killer is			Value	Value	Value
	Peter	Paul	Mary	at t_0	if ¬*Peter*	if ¬*Mary*
BetP*	.281	.302	.417			
$BetP_{\neg Peter}$	.0	.33	.66			
$BetP_{\neg Mary}$	.75	.25	.0			
B1	0	0	24	10	16	0
B2	16	0	0	4.5	0	12
B3	0	4	4	2.875	4	1
Total				17.375		

Table 5.2. Prior (BetP*)and posterior ($BetP_{\neg Peter}$, $BetP_{\neg Mary}$) pignistic probabilities on Ω, outcomes of bets B1, B2 and B3, and values of the three bets before (at t_0) or after observing the outcome of the *X* experiment.

Table 5.3 presents Your net gains in the various possible contexts, i.e. according to what is the outcome of the *X* experiiment and the final outcome on Ω. As can be seen, there is no strategy that can be build at t_0 and such that the Bookie would win whatever the final Ω outcome.

X Outcome If You learn	Ω Outcome	Bets B1	B2	B3	Net Gain	Bookie buys
	Peter	0	16	0	16 - 17.4 = -1.4	
	Paul	0	0	4	4 - 17.4 = -13.4	
	Mary	24	0	4	28 - 17.4 = 10.6	
¬*Peter*			0	4	4 + 16 - 17.4 = 2.6	B1 at $16
¬*Mary*	Peter	0		0	12 - 17.4 = -5.4	B2 at $12
	Paul	0		4	4 + 12 - 17.4 = -1.4	B2 at $12
¬*Mary*	Peter	0	16		16 + 1 - 17.4 = -.4	B3 at $1
	Paul	0	0		1 - 17.4 = -16.4	B3 at $1
¬*Mary*		0			13 - 17.4 = -4.4	B2+B3 at $13

Table 5.3. Net gains of the three outcomes on Ω before and after the observation of the outcome of the *X* experiment and according to the possible Bookie’ strategies for buying back some pending bets.

Table 5.5 presents also the fair prices of the three bets after observing ¬*Peter* or ¬*Mary*.

	You win if killer is			Value	Value	Value
	Peter	Paul	Mary	at t_0	if ¬*Peter*	f ¬*Mary*
BetP*	.125	.50	.375			
$BetP_{\neg Peter}$	.0	.50	.50			
$BetP_{\neg Mary}$	.50	.50	.0			
B1	0	0	24	9	12	0
B2	16	0	0	2	0	8
B3	0	4	4	3.5	4	2
Total				14.5		

Table 5.5. Prior (BetP*)and posterior ($BetP_{\neg Peter}$, $BetP_{\neg Mary}$) pignistic probabilities on Ω, outcomes of bets B1, B2 and B3, and values of the three bets before (at t_0) or after observing the outcome of the *X* experiment.

The next table 5.6. presents Your net gains in the various possible contexts. As in the previous case, the Bookie cannot build a Dutch Book againt You.

X Outcome If You learn	Ω Outcome	Bets B1	B2	B3	Net Gain	Bookie buys
	Peter	0	16	0	16 - 14.5 = 1.5	
	Paul	0	0	4	4 - 14.5 = -10.5	
	Mary	24	0	4	28 - 14.5 = 13.5	
¬*Peter*			0	4	4 + 12 - 14.5 = 1.5	B1 at $12
¬*Mary*	Peter	0		0	8 - 14.5 = -6.5	B2 at $8
	Paul	0		4	4 + 8 - 14.5 = -2.5	B2 at $8
¬*Mary*	Peter	0	16		16 + 2 - 14.5 = 3.5	B3 at $2
	Paul	0	0		2 - 14.5 = -12.5	B3 at $2
¬*Mary*		0			10 - 14.5 = -4.5	B2+B3 at $10

Table 5.6. Net gains of the three outcomes on Ω before and after the observation of the outcome of the *X* experiment and according all possible Bookie' strategies for buying back some pending bets.

To enhance the difference between the Bayesian and the TBM approaches, we present a simplified example that shows that the TBM solutions obtained when ignoring that a given experiment will be run and those derived when knowing about the experiment cannot be achieved simultaneously by some underlying Bayesian analysis.

Suppose a frame Ω = {a, b, c} and let ϖ be the actual value, unknown to You. Let m_0 quantifies Your beliefs that ϖ belongs to the various subsets of Ω (table 6.1). Let E1 and E2 be two posible experiments where E1 tests if ϖ∈ {a,b} or not, and E2 tests if ∈ {b,c} or not. Table 6.1. presents m_0 and the basic belief assignment m_1 and m_2 one would build on Ω after learning the various outcomes of tests E1 and E2, respectively.

	m_0	m_1 {a,b}	m_1 {c}	m_2 {b,c}	m_2 {a}
{a}	.0	.3			1.
{b}	.0			.2	
{c}	.0		1.	.3	
{a,b}	.2	.7			
{a,c}	.3				
{b,c}	.0			.5	
{a,b,c}	.5				

Table 6.1. Basic belief assignments m_0 and m_1 and m_2 one would build on Ω before or after learning the various outcomes of tests E1 and E2.

Context E1: The experimental outcome x is either {a,b} or {c}. Table 6.2. presents the various data.

The prior pignistic probability on the experimental outcome is:

$BetP_{E1} = \Gamma(bel_0, \{\{a,b\},\{c\}\})$.

The pignistic probability on Ω given each of the two possible experimental outcomes is:

$BetP_{\{a,b\}} = \Gamma(bel_{1\{a,b\}}, \{\{a\},\{b\}\})$

$BetP_{\{c\}} = \Gamma(bel_{1\{c\}}, \{\{c\}\})$

where $bel_{1\{a,b\}}$ and $bel_{1\{c\}}$ are the belief functions obtained on Ω after conditioning bel_0 on the E1 outcome.

The pignistic probability on Ω at t_0 is:

$BetP^*_1(y) = BetP_{E1}(x) \,.\, BetP_x(y) \qquad x\in \{\{a,b\},\{c\}\},\ y\in x$

	E1 outcome {a,b}	E1 outcome {c}	
$BetP_{E1}$	.6 = .2 + (.5+.3)/2	.4 = (.5+.3)/2	
Ω	$BetP_{\{a,b\}}$	$BetP_{\{c\}}$	$BetP^*_1$
{a}	.65 = .3 + .7/2		.39
{b}	.35 = .7/2		.21
{c}		1.0	.40

Table 6.2: Context E1: values of the pignistic probabilities ($BetP^*_1$) on Ω and on the experimental outcomes ($BetP_{E1}$).

Context E2: The experimental outcome x is either {b,c} or {a}. Table 6.3. presents the various data.

The prior pignistic probability on the experimental outcome is:

$BetP_{E1} = \Gamma(bel_0, \{\{b,c\},\{a\}\})$.

The pignistic probability on Ω given each of the two possible experimental outcomes is:

$BetP_{\{b,c\}} = \Gamma(bel_{1\{b,c\}}, \{\{b\},\{c\}\})$

$BetP_{\{a\}} = \Gamma(bel_{1\{a\}}, \{\{a\}\})$

where $bel_{1\{b,c\}}$ and $bel_{1\{c\}}$ are the belief functions obtained on Ω after conditioning bel_0 on the E2 outcome.

The pignistic probability on Ω at t_0 is:

$BetP^*_2(y) = BetP_{E2}(x) . BetP_x(y) \quad x \in \{\{b,c\},\{a\}\}, \; y \in x$

	E2 outcome		
	{b,c}	{a}	
$BetP_{E2}$	.5 = (.2+.3+.5)/2	.5 = (.2+.3+.5)/2	
Ω	$BetP_{\{b,c\}}$	$BetP_{\{a\}}$	$BetP^*_2$
{a}		1.0	.50
{b}	.45 = .2 + .5/2		.225
{c}	.55 = .3 + .5/2		.275

Table 6.3: Context E2: values of the pignistic probabilities ($BetP^*_2$) on Ω and on the experimental outcomes ($BetP_{E2}$).

With the TBM analysis, You have only one belief on Ω characterized by m_0 from which You derive Your pignistic probabilities $BetP^*_1$ and $BretP^*_2$ according to which experiment will be perfomred, and You will bet accordingly. One might wonder if the TBM analysis is not just a rephrasing of some equivalent Bayesian analysis.

Tbale 6.4. presents the pignistic probabilities $BetP_0$ obtained from bel_0 on Ω and those derived after learning that a test (E1 or E2) is going to be run. In a Baysians analysis, one could try to assume that both $BetP^*_1$ and $BetP^*_2$ are nothing but the conditional probability measures on Ω given the knowledge about the E1 and E2 tests. But then $BetP^*_0$ should correspond to the marginalization on Ω of these two conditional probability measures. An essential property of this marginalization process is that the result should be numerically between the two conditional values. As seen in table 6.4, $BetP^*_0(\{b\})$ is larger than both $BetP^*_1(\{b\})$ and $BetP^*_2(\{b\})$. So the TBM results cannot be obtained by a Bayesian analysis.

	$BetP^*_0$	$BetP^*_1$	$BetP^*_2$
{a}	.417	.39	.50
{b}	.267	.21	.225
{c}	.317	.40	.275

Table 6.4: Pignistic probabilities before knowing thzat some experiment will be run ($BetP^*_0$) and after learning that experiments E1 or E2 is going to be run ($BetP^*_1$, $BetP^*_2$).

The next Figure summarizes the fact that the TBM and the Bayesians solutions are incompatible.