

# THE APPLICATION OF THE TRANSFERABLE BELIEF MODEL TO DIAGNOSTIC PROBLEMS.

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Short presentation of the most relevant elements of the transferable belief model and its use for problems related to the diagnostic process. These examples illustrate the use of the transferable belief model and in particular of the Generalized Bayesian Theorem.

## **1. Introduction.**

Uncertainty is classically represented by probability functions, and diagnostic in an environment poised by uncertainty is usually handled through the application of the Bayesian theorem that permits the computation of the posterior probability over the diagnostic categories given the observed data from the prior probability over the same categories. We show here that the whole problem admits a similar solution when uncertainty is quantified by belief functions as in the transferable belief model. The classical Bayesian theorem admits a generalization within the transferable belief model (TBM) that we called the Generalized Bayesian Theorem (Smets, 1978, 1981, 1993a).

This theorem seems to have been often overlooked, and the use of conditional belief functions for diagnostic problems neglected. The Generalized Bayesian Theorem (GBT) permits the computation of the conditional belief over the diagnostic classes given an observed data from the knowledge of the set of the conditional beliefs about which data will be observed when the case belongs to a given diagnostic category. Loosely expressed, this inversion theorem permits to pass from a belief on the symptoms given the diseases to a belief on the diseases given the symptoms. We present hereafter four examples of diagnostic process within the TBM, and compared the TBM solution with its obvious contender, the probability solution. The examples are analyzed in detail in order to give a clear understanding of the exact use of the TBM and its GBT. We restrict ourself to 'simple' examples, cases of complex systems and common or dependent causes are not tackled. Our aim is in showing how the classical Bayesian theorem can be extended and applied within the TBM framework.

We first present the needed background knowledge about the TBM (section 2) and the GBT (section 3). Then we present four examples (section 4). We give some comments to explain the conceptual difference between the Bayesian networks and the evidential network (EVN), the two networks used to implement a diagnostic problem within the probability and the belief function framework, respectively (section 5). We then conclude (section 6).

## 2. The transferable belief model.

The transferable belief model (TBM) is a model developed to represent quantified beliefs (Smets and Kennes, 1994). It covers the same domain as the Bayesian - subjectivist probabilities except it is not based on probability functions but on so-called belief functions (Shafer, 1976, Smets, 1988, Smets et Magrez, 1987, Smets 1994). One starts from a finite set of worlds  $\Omega$  called the frame of discernment. One of its worlds, denoted  $\omega_0$ , corresponds to the **actual world**. The agent, denoted You (but it might be a robot, a piece of software), does not know which world in  $\Omega$  corresponds to the actual world  $\omega_0$ . Nevertheless, You have some idea, some opinion about which world might be the actual one. So for every subset  $A$  of  $\Omega$ , You can express the strength of Your opinion that the actual world  $\omega_0$  belongs to  $A$ . This strength is denoted  $\text{bel}(A)$  with  $\text{bel}:2^\Omega \rightarrow [0,1]$ . Extreme values for  $\text{bel}$  denote full belief (1) or no belief at all (0). The larger  $\text{bel}(A)$ , the stronger You believe  $\omega_0 \in A$ . Up to here  $\text{bel}$  shares the same properties as the classical subjective probability measure. The TBM departs from the classical Bayesian approach in that we do not assume the additivity encountered in probability theory. For instance, we do not assume that  $\text{bel}(A) = 0$  implies that  $\text{bel}(\bar{A}) = 1$ . In fact  $\text{bel}(A) = \text{bel}(\bar{A}) = 0$  is even possible. The additivity property that characterizes the probability functions is replaced by inequalities like:

$$\text{bel}(A \cup B) \geq \text{bel}(A) + \text{bel}(B) - \text{bel}(A \cap B) \quad (2.1)$$

In the TBM, one assumes that  $\text{bel}$  is a capacity monotone of order infinite, i.e.,  $\text{bel}$  satisfies the following inequalities:

$$\forall n > 1, \forall A_1, A_2, \dots, A_n \subseteq \Omega, \\ \text{bel}(A_1 \cup A_2 \cup \dots \cup A_n) \geq \sum_i \text{bel}(A_i) - \sum_{i > j} \text{bel}(A_i \cap A_j) \dots - (-1)^n \text{bel}(A_1 \cap A_2 \cap \dots \cap A_n) \quad (2.2)$$

As such, the meaning of these inequalities is not obvious except when  $n = 2$ , in which case one gets relation (2.1). These inequalities generalize the idea that Your belief that the actual world belongs to  $A \subseteq \Omega$  can be larger than the sum of the beliefs You give to the elements of a partition of  $A$ .

### *Basic belief assignment.*

The understanding of the inequalities (2.2) is clarified once the concept of a **basic belief assignment** (bba) is introduced. Related to  $\text{bel}$ , one can define its so-called Möebius transform, denoted  $m$  and called a basic belief assignment in the present context. Let  $m:2^\Omega \rightarrow [0,1]$  where  $m(A)$  is called the **basic belief mass** (bbm) given to  $A \subseteq \Omega$ . The value  $m(A)$  represents that part of Your belief that supports  $A$  - i.e., the fact that the actual world  $\omega_0$  belongs to  $A$  - without supporting any more specific subset, by lack of adequate information. When You learn that  $\omega_0$  belongs to  $A$ , and You know nothing else about the value of  $\omega_0$ , then some part of Your belief will be given to  $A$ , but no subset of  $A$  will get any positive support. In

that case, You would have  $m(A) > 0$  and  $m(B) = 0$  for all  $B \subseteq A$  and  $B \neq A$ , a property that could not be satisfied by a probability measure.

### **Belief functions.**

The bba  $m(A)$  does not in itself quantify Your belief, denoted  $\text{bel}(A)$ , that the actual world  $\omega_0$  belongs to  $A$ . Indeed, the bba  $m(B)$  given to any subset  $B$  of  $A$  also supports that  $\omega_0 \in A$ . Hence, the belief  $\text{bel}(A)$  is obtained by summing all the bba  $m(B)$  for  $B \subseteq A$ . The only exception concerns  $m(\emptyset)$ . In some theories (Shafer, 1976), one assumes  $m(\emptyset) = 0$ . In the TBM, such a requirement is not assumed. Nevertheless  $m(\emptyset)$  is not included in  $\text{bel}(A)$ . Indeed, the empty set  $\emptyset$  is a subset of  $A$  but also of its complement  $\bar{A}$ . We want  $\text{bel}$  to denote the amount of belief given specifically to the fact that the actual world  $\omega_0$  belongs to  $A$ , hence we do not include in  $\text{bel}(A)$  those basic belief masses that support also  $\bar{A}$ , i.e.,  $m(\emptyset)$ . We have:

$$\begin{aligned} \text{bel}(A) &= \sum_{\emptyset \neq B \subseteq A} m(B) \quad \forall A \subseteq \Omega, A \neq \emptyset & (1.3) \\ \text{bel}(\emptyset) &= 0. \end{aligned}$$

The function  $\text{bel}$  so defined satisfies the inequalities (2.2). Thanks to the natural interpretation that can be given to the basic belief masses, the meaning of the inequalities (2.2) becomes somehow clearer. The originality of this approach comes from the non null masses that may be given to non singletons of  $\Omega$ . Indeed, when  $m(A) = 0$  for all  $A \subseteq \Omega$  with  $|A| > 1$  ( $|A|$  denotes the number of elements in  $A$ ), then the function  $\text{bel}$  is a probability function, and the TBM reduces itself to the Bayesian theory. Of course, in general, this requirement is not satisfied in the TBM, and thus the bba  $m(A)$  given to those  $A$  with  $|A| > 1$  give it its specificity.

Those subsets  $A$  of  $\Omega$  which bba  $m(A)$  is strictly positive are called the **focal elements** of the bba (or of  $\text{bel}$ ). A **simple support function** is a belief function that has only two focal elements,  $\Omega$  and one subset  $A$  of  $\Omega$ .

The advantage of the TBM over the classical Bayesian approach resides in its large flexibility, its ability to represent more elaborated beliefs and this includes its ability to represent every state of partial beliefs, up to the **state of total ignorance** (Dubois et al., 1996). In the TBM, total ignorance is represented by the so-called **vacuous belief function**, i.e., a belief function such that  $m(\Omega) = 1$ ,  $m(A) = 0$  for all  $A$  with  $A \subseteq \Omega$ ,  $A \neq \Omega$ . Hence  $\text{bel}(\Omega) = 1$  and  $\text{bel}(A) = 0$  for all  $A$  with  $A \subseteq \Omega$ ,  $A \neq \Omega$ . It expresses that all You know is that  $\omega_0 \in \Omega$ . The representation of total ignorance in probability theory is hard to achieve adequately, most proposed solutions being doomed to contradictions. With the TBM, we can of course represent every state of belief, full ignorance, partial ignorance, probabilistic-additive beliefs, or even full belief ( $m(A) = 1$  corresponds to  $A$  is certain). Such an expressivity power explains the interest of the TBM.

### **Related functions.**

Related to  $\text{bel}$  and  $m$ , one defines a plausibility function  $\text{pl}: 2^\Omega \rightarrow [0,1]$  so that:

$$\text{pl}(A) = \text{bel}(\Omega) - \text{bel}(\bar{A}) \quad \text{for all } A \subseteq \Omega$$

$$\text{or } pl(A) = \sum_{X \subseteq \Omega: X \cap A \neq \emptyset} m(X) \quad \text{for all } A \subseteq \Omega.$$

Both relations are equivalent. The value  $pl(A)$  represents the amount of belief that possibly (might) support that the actual world  $\omega_0$  belongs to  $A$ , whereas  $bel(A)$  represents the amount of belief that necessarily (do) support that the actual world  $\omega_0$  belongs to  $A$ .

Two other functions related to  $bel$  are also defined: the commonality function  $q: 2^\Omega \rightarrow [0,1]$  and the b-function  $b: 2^\Omega \rightarrow [0,1]$  with:

$$q(A) = \sum_{X \subseteq \Omega: A \subseteq X} m(X) \quad \text{for all } A \subseteq \Omega.$$

$$b(A) = bel(A) + m(\emptyset) = \sum_{X \subseteq \Omega: X \subseteq A} m(X) \quad \text{for all } A \subseteq \Omega.$$

Their major interest will appear when conditioning and combination will be introduced.

### Least Commitment Principle.

In probability theory, probability functions can be ordered by their ‘information content’, what is usually achieved by the entropy  $Entr$  where the entropy of the probability measure  $P$  is:

$$Entr(P) = - \sum_{x \in \Omega} P(\{x\}) \cdot \log(P(\{x\}))$$

The larger the entropy, the smallest the ‘informativeness’ of  $P$ . The entropy is often used to select the least informative probability function (hence the probability function with the largest entropy) in a set of probability functions. A similar criteria is defined within the TBM. Suppose two plausibility functions  $pl_1$  and  $pl_2$ , both on  $\Omega$ . If for all  $A \subseteq \Omega$ ,  $pl_1(A) \geq pl_2(A)$ , then we say that  $pl_1$  is less committed than  $pl_2$ . Indeed, the least committed plausibility function is the vacuous one:  $pl(A) = 1$  for all  $A \subseteq \Omega$ . The inequality can be replaced identically by:

$$b_1(A) \leq b_2(A) \quad \text{for all } A \subseteq \Omega,$$

or  $bel_1(A) \leq bel_2(A) \quad \text{for all } A \subseteq \Omega$ , only if  $m_1(\emptyset) = m_2(\emptyset)$ ,

To be least committed means that each subset gets less or equal support. Such property can be used to define a partial order on the set of belief functions. Whenever You must choose between several belief functions, and there is no extra reason to prefer any one of them, You can then evoke the Principle of Least Commitment in order to select the least committed belief function. The Principle can be paraphrased by ‘never give more support than necessary’.

### Conditioning.

The dynamics of belief, hence how beliefs change when new information becomes known to You, is described by several rules. The most important is the **conditional rule** that translates the impact of learning for sure that the actual world  $\omega_0$  belongs to  $X$ . Let  $m_X$ ,  $bel_X$ ,  $pl_X$ ,  $b_X$  and  $q_X$  be the various conditional functions obtained after conditioning the initial belief function on  $\omega_0 \in X$ . The mass  $m(A)$  initially supporting that  $\omega_0 \in A$  supports that  $\omega_0 \in A \cap X$  after learning that  $\omega_0 \in X$  for sure. So:

$$m_X(A) = \sum_{B \subseteq \bar{X}} m(A \cup B) \quad A \subseteq X$$

what implies that:

$$\begin{array}{ll}
 \text{bel}_X(A) = \text{bel}(A \cup \bar{X}) - \text{bel}(\bar{X}) & \text{for all } A \subseteq X \\
 \text{pl}_X(A) = \text{pl}(A \cap X) & \text{for all } A \subseteq X \\
 \text{b}_X(A) = \text{b}(A \cup \bar{X}) & \text{for all } A \subseteq X \\
 \text{and } \text{q}_X(A) = \text{q}(A) & \text{if } A \subseteq X \\
 & = 0 \quad \text{otherwise.}
 \end{array}$$

This rule of conditioning is called Dempster's rule of conditioning, except for a possible normalization that we don't apply automatically in the TBM. This rule plays the role, within the TBM, of the rule of conditioning described in probability theory.

### **Combination.**

Another rule concerns the combination of two belief functions. Suppose You collect two 'distinct' pieces of evidence  $Ev_1$  and  $Ev_2$  produced by two sources of information. Let  $\text{bel}_1$  and  $\text{bel}_2$  be the belief functions induced by each piece of evidence taken individually. Then You can combine these two belief functions in at least two ways: conjunctively or disjunctively, by what we mean that You want to build Your belief given You know that *both* sources of information are fully reliable (conjunctive combination), or given You know only that *at least one* of the two sources is reliable (disjunctive combination). The resulting belief functions  $\text{bel}_{1 \wedge 2}$  and  $\text{bel}_{1 \vee 2}$  are obtained from the next relations.

#### **Conjunctive combination:**

$$m_{1 \wedge 2}(A) = \sum_{X, Y \subseteq \Omega: X \cap Y = A} m_1(X) m_2(Y) \quad \text{for all } A \subseteq \Omega$$

$$q_{1 \wedge 2}(A) = q_1(A) q_2(A) \quad \text{for all } A \subseteq \Omega.$$

#### **Disjunctive combination:**

$$m_{1 \vee 2}(A) = \sum_{X, Y \subseteq \Omega: X \cup Y = A} m_1(X) m_2(Y) \quad \text{for all } A \subseteq \Omega$$

$$b_{1 \vee 2}(A) = b_1(A) b_2(A) \quad \text{for all } A \subseteq \Omega.$$

The interest of the  $q$  and  $b$  functions becomes obvious as they greatly simplified the computation required to compute the combinations of two belief function. The conjunctive rule is Dempster's rule of combination, but unnormalized.

The meaning of 'distinct' for two pieces of evidence has been left undefined. It lacks rigorous definition. Intuitively it means the absence of any relation, of any link between the two sources. It can also be understood as meaning that the belief function  $\text{bel}_2$  induced by the second source is not influenced by the knowledge of the belief function  $\text{bel}_1$  induced by the first source (knowing  $\text{bel}_1$  does not interfere with the assessment of  $\text{bel}_2$ ) and vice versa (Smets, 1992b) .

### **Decision process.**

A normative theory for **decision process** has been fully developed within the TBM (Smets, UAI, Smets and Kennes, 1994). Given a belief function, we generate a probability function that must be used to make decision by maximizing expected utilities. It requires first the construction of the betting frame, i.e., a list of alternatives on which the bet must be made. Let  $Bf$  denoted the betting frame. The granularity of  $Bf$  is such that if by necessity two alternatives are not distinguishable from a consequence-utility point of view, than they are pooled into the same granule. Once the betting frame  $Bf$  is determined, the initial plausibility function  $pl_0$  defined on  $\Omega$  is transformed into a plausibility function  $pl_1$  on  $Bf$ , so that:

$$pl_1(X) = pl_0(Y) \text{ for all } X \subseteq Bf \text{ and } Y \text{ is the smallest subset of } \Omega \text{ so that } X \subseteq Y^1.$$

The bba  $m_1$  on  $Bf$  is derived from  $pl_1$ . The bba  $m_1$  is then transformed by the so-called **pignistic transformation** into the **pignistic probabilities**  $BetP: 2^\Omega \rightarrow [0,1]$  with:

$$BetP(A) = \sum_{X \subseteq Bf, X \neq \emptyset} \frac{m_1(X)}{1 - m_1(\emptyset)} \frac{\#(A \cap X)}{\#(X)} \quad \text{for all } A \subseteq Bf, A \neq \emptyset,$$

and  $\#(Y)$  is the number of granules of the betting frame  $Bf$  in  $Y$ . By construction, the pignistic probability function  $BetP$  is a probability function, but we qualified it as pignistic to avoid the error that would consist in considering this probability function as representing Your beliefs. Your beliefs are represented by  $bel_0$ , and  $BetP$  is just the additive measure needed to compute expected utilities when decision must be made (Savage, 1954). The justification of the proposed transformation is given in Smets and Kennes (1994). Dutch Books cannot be built against a user of the TBM, even though conditioning is not achieved by the classical probability rule (Smets, 1993c).

### **3. Generalized Bayesian Theorem.**

In classical probability theory, the Bayesian theorem is the major theorem that underlies the diagnostic process. Suppose a domain  $\Theta$  of possible diseases (or faults, categories, classes...) and a set  $X$  of symptoms (or observables, data, ...). The sets  $\Theta$  and  $X$  are assumed to be finite. Let  $P(x | \theta)$  be known for every  $x \in X$  and every  $\theta \in \Theta$ . It is the distribution function of the symptom  $x$  in the disease category  $\theta$ . Let  $P_0$  be Your a priori belief about which disease category the patient under consideration belongs to. Then given You observe symptom  $x$ , Your a posteriori belief about the disease categories to which the patient belongs to is computed from the Bayesian theorem:

$$P(\theta | x) = \frac{P(x | \theta) P_0(\theta)}{\sum_{\theta_i \in \Theta} P(x | \theta_i) P_0(\theta_i)} \quad \text{for all } \theta \in \Theta. \quad (3.1)$$

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<sup>1</sup>This notation is quite loose, the rigorous notation is given in Smets and Kennes (1994).

This theorem is considered both as a learning theorem as an inversion theorem. In the first case, it permits the computation of the a posteriori probability given the a priori probability and a learned fact. In the second case, it permits to compute the probability on  $\Theta$  given  $x$  from the probability on  $X$  given  $\theta$ . Adapting the wording to fit with faults diagnostic or any other form of diagnosis (i.e., of classification process) is straightforward: symptoms are the observed data, the facts, and the diseases are the classes into which the case must be allocated.

We have generalized this inversion theorem within the TBM framework (Smets, 1978, 1981, 1993a). Your belief about which fact  $x$  can be observed in each class  $\theta \in \Theta$  is represented by the belief function  $\text{bel}(x | \theta)$  for all  $x \subseteq X$ . We also assume that Your a priori belief  $\text{bel}_0$  defined on  $\Theta$  about the class to which Your case belongs is vacuous (total a priori ignorance). Then the a posteriori belief  $\text{bel}_1$  and its related functions on  $\Theta$  about the class to which Your case belongs given You observe the fact  $x$  (that  $x$  is an element or a subset of  $X$  is irrelevant here) are given by: for  $\theta \subseteq \Theta$  and  $x \subseteq X$ ,

$$\begin{aligned}
 b_1(\theta | x) &= \prod_{\theta_i \in \bar{\theta}} b(\bar{x} | \theta_i) \\
 \text{bel}_1(\theta | x) &= b_1(\theta | x) - b_1(\emptyset | x) \\
 \text{pl}_1(\theta | x) &= 1 - \prod_{\theta_i \in \theta} (1 - \text{pl}(x | \theta_i)) \\
 \text{q}_1(\theta | x) &= \prod_{\theta_i \in \theta} \text{pl}(x | \theta_i)
 \end{aligned} \tag{3.2}$$

These relations are called the Generalized Bayesian Theorem (GBT). They can be adapted for the case where You have some non vacuous a priori belief  $\text{bel}_0$  on  $\Theta$ : You just combine conjunctively on  $\Theta$  the a priori  $\text{bel}_0$  with the function  $\text{bel}_1$  obtained by the GBT.

The belief induced on  $\Theta$  can identically be obtained by construction on  $\Theta$  a simple support function for each conditional belief function  $\text{bel}(x | \theta)$ ,  $\theta \in \Theta$ , received initially, and combining them conjunctively. Suppose the observed fact is  $x \subseteq X$ . For each  $\theta \in \Theta$ , construct the simple support function with the bba given by  $m(\theta | x) = 1 - b(\bar{x} | \theta)$  and  $m(\bar{\theta} | x) = b(\bar{x} | \theta)$ . Combining conjunctively the simple support functions so obtained with each  $\theta \in \Theta$  produces a belief function on  $\Theta$  that is equal to the one obtained by the GBT (3.2).

The GBT has two major interests when compared with the Bayesian theorem.

- 1) We can assume that Your a priori belief over  $\Theta$  is vacuous, hence we avoid all the troubles related to the choice of an adequate a priori probability function. This choice is often critical and leads to major criticism against the use of the Bayesian theorem, what led regularly to its plain rejection.
- 2) Besides, the GBT can also cope with a situation hardly manageable with classical probability theory but still very useful. Suppose  $\Theta$  is not exhaustive because You forgot some possible classes (because of limited understanding, or pure ignorance...). The problem is solved by

creating an extra class. Let us denote it  $\theta_\omega$ . It can be understood as “the still unknown disease class”, “the forgotten alternatives”, “the unconsidered failures”... In order to apply the GBT, You must provide Your belief about which fact  $x$  would be observed if the failure belongs to the category  $\theta_\omega$ . Within the GBT, such a belief is described by the vacuous belief function. Indeed, what do You know about  $X$  for failures that belongs to the set of unlisted causes? Such flexibility is not achievable in probability theory as total ignorance cannot be represented adequately. Within the TBM, computing the a posteriori belief over  $\Theta$  is still straightforward, the presence of a vacuous belief function for  $\theta_\omega$  does not alter the computation. After having introduced  $\theta_\omega$ , it is even possible to compute the a posteriori belief that the case belongs to the  $\theta_\omega$  class.

*The case of two facts.*

An important property of the GBT is the following. Suppose two finite sets  $X$  and  $Y$ , and a set of classes  $\Theta$ . Let  $bel_X(.|\theta)$  and  $bel_Y(.|\theta)$  represent Your beliefs about which fact in  $X$  and  $Y$  would be observed, respectively, if You knew that the case belongs to class  $\theta \in \Theta$ . Suppose You observe the facts  $x \subseteq X$  and  $y \subseteq Y$ . With each of them individually, You can build Your belief on  $\Theta$  by applying the GBT. Let  $bel(.|x)$  and  $bel(.|y)$  be these two belief functions. Then You build Your belief  $bel(.|x,y)$  on  $\Theta$  given You have observed both facts by combining these two belief functions conjunctively.

There is another way to build this final belief function on  $\Theta$ . You first build the observation space  $X \times Y$  and construct Your belief about which pair of fact You will observed. In probability theory, it is often assumed that the two symptoms are conditionally independent, in which case the conditional probability to observed the pair  $(x,y)$  when the case belong to class  $\theta$  is the product of the conditional probabilities to observed each of them individually when the case belongs to class  $\theta$ :

$$P(x,y | \theta) = P(x | \theta).P(y | \theta) \quad \text{for all } x \subseteq X, y \subseteq Y, \theta \in \Theta.$$

This concept is extended into the TBM framework where independence is called **Cognitive Independence**. Two variables  $X$  and  $Y$  are Cognitive Independent if the ratio of the plausibilities on one of them does not depend on the value of the other variable:

$$\frac{pl_X(x_1 | y)}{pl_X(x_2 | y)} = \frac{pl_X(x_1)}{pl_X(x_2)} \quad \forall x_1, x_2 \subseteq X, \forall y \subseteq Y,$$

where the index of the plausibility functions indicate their domains. It implies that:

$$pl_{X \times Y}(x,y) = pl_X(x) pl_Y(y) \quad \forall x \subseteq X, \forall y \subseteq Y.$$

The Cognitive Independence concept can be extended in a straightforward manner when the plausibility functions are conditional plausibility functions. If the two variables  $X$  and  $Y$  are cognitively independent in each context  $\theta$ , for all  $\theta \in \Theta$ , then they satisfy the **Conditional Cognitive Independence** property:

$$pl_{X \times Y}(x,y | \theta) = pl_X(x | \theta) pl_Y(y | \theta) \quad \forall x \subseteq X, \forall y \subseteq Y, \forall \theta_i \in \Theta.$$



From this plausibility function  $pl_{X \times Y}(\cdot | \theta)$ , You can apply the GBT in order to derive Your beliefs  $bel^*(\cdot | x, y)$  on  $\Theta$  given the observation  $(x, y)$ . It happens that  $bel^*(\cdot | x, y)$  is equal to  $bel(\cdot | x, y)$  previously derived. This equality is essential. It shows that the belief obtained from 1) the conjunctive combination of the two beliefs on  $\Theta$  obtained by the GBT and 2) the belief obtained from the GBT when applied to the joint observations are equal as it should be.

We have shown that the GBT is the only inversion theorem that satisfies this property when one requires that the plausibility  $pl(x | A)$  for  $A \subseteq \Theta$  is a function of the terms  $pl(x | \theta)$  and  $pl(\bar{x} | \theta)$  for  $\theta \in A$  (Smets, 1993a).

The GBT has also been derived independently by Appriou (1991). The author requires that the inversion theorem degenerates into the classical Bayesian theorem when all involved belief functions are probability functions, that the equality we just derived in the case of Conditional Cognitive Independence holds, and that each individual conditional belief function on  $X$  given  $\theta \in \Theta$  induces a belief function on  $\Theta$  where the only focal elements are  $\theta$ ,  $\bar{\theta}$  and  $\Theta$ .

Other formulas have been proposed in order to generalize the Bayesian theorem. None satisfies the property we described in the case of Conditional Cognitive Independence. This is not surprising as, under very natural requirements, the GBT is the only transformation that satisfies this property (Smets, 1993a).

#### **4. Illustrative examples.**

We illustrate the use of the TBM with four examples. The first, the breakable sensors, enhances the difference between the TBM and the Bayesian approach. The second concerns a classical fault diagnosis and enhance the possibility to build a priori an optimal strategy. The third is a classical medical diagnosis that illustrates the use of the GBT. The fourth concerns the location of some radio-active leakage and shows the power of the GBT as it can be extended to cope with a totally unknown location.

##### **4.1. The breakable sensors.**

Suppose You are a new technician and You must check the temperature of a process. To do this, You receive five sensors. The sensors are of two types: one of them (sensor 1) is a reliable but expensive sensor, and the four others (sensors 2 to 5) are cheaper but less reliable sensors. Each sensor is applied to determine the temperature of the process - a temperature that can be only 'Hot' or 'Cold'. When the sensor is in good working conditions, if the temperature is hot, the sensor's light is red and if the temperature is cold, the sensor's light is blue. Each sensor is made of a thermometer and a device that turns on the blue-red light according to the temperature reading. Unfortunately, the thermometer may be broken: the probability that the expensive

sensor is broken is 0.01, whereas the probability that each of the four cheaper sensors is broken is 0.12. The event that the sensors are broken are independent.

The only information known to You is what is written on the five boxes containing the sensors. For each of the four cheap one, You read: "Warning: the thermometer included in this sensor may be broken. The probability that it is broken is 12%. When the thermometer is not broken, the sensor is a perfectly reliable detector of the temperature situation. When the thermometer is not broken, red light means the temperature is hot, blue light means the temperature is cold. When the thermometer is broken, the sensor answer is unrelated to the temperature". For the expensive one, You read the same text, except the 12% is replaced by 1%.

You are a new technician and have never seen these sensors before. You know nothing about them except the warnings written on the boxes. You use the four sensors as ordered. The red light goes on for the expensive sensor and for one cheap sensor, and the blue one goes on for the three cheap sensors. How do You assess the temperature status? What is Your belief that the temperature status is hot or cold? And supposing You must act differently according to the temperature of the process, how would You decide?

**TBM analysis.**

We first analysis the problem in the TBM framework. Table 4.1.1 presents the sensor's reliabilities. Table 4.1.2 presents the focal sets on the product space Light x Temperature x Sensor's status. With sensor 1, the bbm are:

$$m( \{ (Red, Hot, Working), (Blue, Cold, Working) \} ) = .99$$

and  $m( \{Red, Blue\} \times \{Hot, Cold\} \times \{Broken\} ) = .01.$

The bbm translates the fact that with probability .99 the sensor is in working condition in which case Red = Hot and Blue = Cold, and with probability .01 it is broken in which case there is no relation between the temperature's status and the sensor's answer. Table 4.1.3 presents the bba induced on the temperature's status for sensor 1 when the sensor's answer is red and blue, respectively.

	Reliability
Sensor 1	0.99
Sensors 2 to 5	0.88

**Table 4.1.1:** Reliability of the five sensors.

Sensor's answer	Red	Red	Blue	Blue	
Water temperature	Hot	Cold	Hot	Cold	bbm
Working Cond.	1	-	-	1	0.99
Broken	1	1	1	1	0.01

**Table 4.1.2:** TBM analysis: for sensor 1, the two focal elements with their bbm (the 1's indicate those elements in each focal element).

If Sensor's answer =	Red	Blue
∅	0.00	0.00
Hot	0.99	0.00
Cold	0.00	0.99
Hot or Cold	0.01	0.01

**Table 4.1.3:** TBM analysis: for sensor 1, the bba on the temperature given the sensor's light is red or blue, respectively.

Suppose Sensors 1 and 2's answers are Red and Sensors 3 to 5's answers are Blue. Each sensor induces a bba on the temperature status, which bbm are given in table 4.1.4. These bba are then combined by the conjunctive combination rule (rightmost column). It shows a strong contradiction ( $m(\emptyset) = .997$ ) as expected in this case. Indeed two sensors support hot and three support cold, and each sensor is quite reliable.

Sensor	1	2	3	4	5	
Reliability	0.99	0.88	0.88	0.88	0.88	
Sensor's Answer	Red	Red	Blue	Blue	Blue	
	bba1	bba2	bba3	bba4	bba5	Comb.
∅	0.00	0.00	0.00	0.00	0.00	0.9971
Hot	0.99	0.88	0.00	0.00	0.00	0.0017
Cold	0.00	0.00	0.88	0.88	0.88	0.0012
Hot or Cold	0.01	0.12	0.12	0.12	0.12	0.0000

**Table 4.1.4:** TBM analysis: for each sensor, the bba on the temperature given the sensor's answer. The last column presents the bba obtained by the conjunctive combination of the five bba (Comb. column).

	Sensor	Observation		
	1	2	3	4
	1	Red	Red	Red
	2	Blue	Red	Red
	3	Blue	Blue	Red
	4	Blue	Blue	Blue
	5	Blue	Blue	Blue
	Case	1	2	3
TBM	Hot	0.020	0.590	0.990
BetP(Temp Obser.)	Cold	0.980	0.410	0.010

**Table 4.1.5:** For three sets of observations (cases 1 to 3), values of the pignistic probability BetP on the temperature status given the temperature status.

Three sets of observations are considered (table 4.1.5). Data of Table 4.1.4 correspond to those of the second example. The pignistic probability BetP on the temperature's status are computed from the bba obtained after conjunctively combining the five bba obtained from each of the five sensors. So for the second case (table 4.1.4),

$$\begin{aligned} \text{BetP}(\text{Hot} | \text{data}) &= (m(\text{Hot}) + m(\text{Hot or Cold})/2) / (1 - m(\emptyset)) \\ &= (.0017 + .0000/2) / (1 - .9971) = .5902 \end{aligned}$$

The other values of BetP are presented in table 4.1.5. So in case 1, cold is highly probable temperature (only sensor 1 supports Hot), hot is a little more probable in the second case

(Sensor 1 and 2 support Hot), and highly probable in the third cases (Sensor 1 to 3 support Hot). For the first case, it was not obvious that the joint opinion of four ‘cheap’ sensors would beat the opinion of one ‘good’ sensor, but it seems surely acceptable. For the third case, the opinions of the ‘cheap’ sensors balance each other, and the ‘good’ one puts the balance toward Hot. For the second case, we just discover that the joint opinion of one ‘good’ and one ‘cheap’ sensors is stronger than the joint opinion of three ‘cheaper’ sensors.

**Probabilistic analysis.**

We then proceed with the probabilistic analysis. Table 4.1.6 presents, for each sensor, the conditional probability on the sensor’s answer given the temperature’s status. It is based on the reliability value and the extra assumption that the sensor’s answer will be red or blue with equiprobability (.5) when the sensor is broken. So, for sensor 1, the probability of Red when Hot is the probability that the sensor is in working condition (.99) plus the probability to be broken (.01) multiplied by the probability of a Red answer when the sensor is broken (.5). The origin of this data comes from classical probability theory:

$$P(\text{Red} | \text{Hot}) = P(\text{Red} | \text{Hot, Working}) P(\text{Working} | \text{Hot}) + P(\text{Red} | \text{Hot, Broken}) P(\text{Broken} | \text{Hot})$$

One have  $P(\text{Red} | \text{Hot, Working}) = 1,$   
 $P(\text{Working} | \text{Hot}) = P(\text{Working}) = .99$   
 $P(\text{Red} | \text{Hot, Broken}) = P(\text{Red} | \text{Broken}) = .5$   
 $P(\text{Broken} | \text{Hot}) = P(\text{Broken}) = .01$

Hence  $P(\text{Red} | \text{Hot}) = .99 + .5 \times .01 = .995.$

Temperature Sensor	Hot 1	Hot 2-3-4-5	Cold 1	Cold 2-3-4-5
Reliability	0.99	0.88	0.99	0.88
Red	0.995	0.940	0.005	0.060
Blue	0.005	0.060	0.995	0.940

**Table 4.1.6:** Probabilistic analysis: conditional probability functions on the sensor’s answer given the temperature’s status, for each sensor.

The equiprobability assumption is based on a classical symmetry principle: “As far as there is no more reason to believe that the answer will be red than blue when the sensor is broken, the two alternatives receive equal probabilities”. If You had some reason to put other values, the whole example could be just as well redone with new data so that the embarrassing conclusions toward which we are proceeding would still be derived. In order to compute the a posteriori probability on the temperature’s status given the observed data, we need the likelihood on the temperature given the data, denoted  $l(\text{temperature’s state} | \text{observed data})$ . This likelihood is equal to the probability to observe the data given the temperature. Let the observed data = (Red<sub>1</sub>, Red<sub>2</sub>, Blue<sub>3</sub>, Blue<sub>4</sub>, Blue<sub>5</sub>) where the numbers indicate the sensor producing the information.

We must compute  $l(\text{Hot} | \text{observed data}) = P(\text{observed data} | \text{Hot})$ .

We use the following notation:

$x_i$  is the observed answer of sensor  $i$  where  $x_i \in \{\text{Red}_i, \text{Blue}_i\}$

$s_i$  is the working / broken status of sensor  $i$ , where  $s_i \in \{w, b\}$

Let  $\text{rel}$  denote one combination of sensor's status: e.g.,  $\text{rel} = (w_1, b_2, w_3, w_4, w_5)$  means that sensor 2 is broken, and the others are in working conditions.

One has:

$$\begin{aligned} P(\text{observed data} \mid \text{Hot}) &= P(x_1, x_2, x_3, x_4, x_5 \mid \text{Hot}) \\ &= \sum_{\text{rel}} P(x_1, x_2, x_3, x_4, x_5 \mid \text{Hot}, s_1, s_2, s_3, s_4, s_5) P(s_1, s_2, s_3, s_4, s_5 \mid \text{Hot}) \end{aligned}$$

where  $\text{rel}$  denotes every combination of sensor's status.

Given the independence assumptions between the working conditions of the five sensors among themselves and with the temperature's status, one has:

$$P(s_1, s_2, s_3, s_4, s_5 \mid \text{Hot}) = P(s_1, s_2, s_3, s_4, s_5) = \prod_i P_i(s_i)$$

where  $P_i(s_i)$  is the probability that sensor  $i$  is in status  $s_i$ .

$$\begin{aligned} \text{Then } P(x_1, x_2, x_3, x_4, x_5 \mid \text{Hot}, s_1, s_2, s_3, s_4, s_5) &= \\ &= P(x_1 \mid \text{Hot}, s_1, s_2, s_3, s_4, s_5, x_2, x_3, x_4, x_5) \cdot P(x_2, x_3, x_4, x_5 \mid \text{Hot}, s_1, s_2, s_3, s_4, s_5) \\ &= P(x_1 \mid \text{Hot}, s_1) \cdot P(x_2, x_3, x_4, x_5 \mid \text{Hot}, s_2, s_3, s_4, s_5). \end{aligned}$$

This results from the fact that the status of sensors 2 to 5 is irrelevant to sensor 1's status once you know its status and the temperature status, and sensor 1's status is irrelevant to the other sensors' answer. Iterating this equality leads to:

$$P(x_1, x_2, x_3, x_4, x_5 \mid \text{Hot}, s_1, s_2, s_3, s_4, s_5) = \prod_i P(x_i \mid \text{Hot}, s_i)$$

$$\text{Hence: } P(x_1, x_2, x_3, x_4, x_5 \mid \text{Hot}) = \sum_{\text{rel}} \prod_i (P(x_i \mid \text{Hot}, s_i)) \cdot P_i(s_i).$$

It can then be shown that this relation is also equal to:

$$\begin{aligned} P(x_1, x_2, x_3, x_4, x_5 \mid \text{Hot}) &= \prod_i (P(x_i \mid \text{Hot}, s_i=w) \cdot P_i(s_i=w) + P(x_i \mid \text{Hot}, s_i=b) \cdot P_i(s_i=b)) \\ &= \prod_i P(x_i \mid \text{Hot}) \end{aligned}$$

Table 4.1.6 presents the values of  $P(x_i \mid \text{Hot})$  and  $P(x_i \mid \text{Cold})$ , and table 4.1.7 presents the values of  $P(\text{observed data} \mid \text{Hot})$  and  $P(\text{observed data} \mid \text{Cold})$  for the three cases presented in table 4.1.5.

P(obsev. Temp.)	Case		
	1	2	3
Hot	0.000013	0.000202	0.003165
Cold	0.003904	0.000249	0.000016

**Table 4.1.7:** Values of the probability of the observed data given the temperature's status for the three cases presented in table 4.1.5.

### *Comparing the two approaches.*

In order to compare the two approaches, the TBM and the probability approach, we present the results as odds. The odds for an hypothesis  $H$  is the ratio  $P(H) / (1-P(H))$  where  $P(H)$  is the

probability of the hypothesis. It is just another way to present probabilities, but convenient in this case. We have:

$$\frac{P(\text{Hot} \mid \text{data})}{P(\text{Cold} \mid \text{data})} = \frac{P(\text{data} \mid \text{Hot}) P(\text{Hot})}{P(\text{data} \mid \text{Cold}) P(\text{Cold})} = \frac{P(\text{data} \mid \text{Hot})}{P(\text{data} \mid \text{Cold})} \frac{P(\text{Hot})}{P(\text{Cold})}$$

So the posterior odds (after conditioning on the data) is equal to the prior odds (before considering the data) multiplied by a coefficient that is the ratio of the likelihoods  $l(\text{Hot} \mid \text{data})$  and  $l(\text{Cold} \mid \text{data})$ . The value of the coefficient is independent of the prior probabilities, hence its interest. A value less than one implies that the a posteriori probability for Hot is smaller than the a priori probability for Hot, and a value larger than one implies that the a posteriori probability for Hot is larger than the a priori probability for Hot.

In the TBM, similar relations are derived when Your a priori belief on the temperature's status is represented by a probability function, as it is assumed in the probability approach. One has:

$$\frac{\text{Bet}P(\text{Hot} \mid \text{data})}{\text{Bet}P(\text{Cold} \mid \text{data})} = \frac{\text{pl}(\text{data} \mid \text{Hot})}{\text{pl}(\text{data} \mid \text{Cold})} \frac{\text{Bet}P(\text{Hot})}{\text{Bet}P(\text{Cold})}$$

In the TBM, the plausibility plays the role of the likelihood, a link that was already encountered in many other contexts, and clearly illustrates by the relation:

$$\text{pl}(\text{Hot} \mid \text{data}) = \text{pl}(\text{data} \mid \text{Hot}).$$

In probability theory, one writes:

$$l(\text{Hot} \mid \text{data}) = P(\text{data} \mid \text{Hot}).$$

where the concept of likelihood is needed as  $l$  is not a probability function ( $l(\text{Hot} \mid \text{data}) + l(\text{Cold} \mid \text{data}) \neq 1$ ). In the TBM, such distinction is not needed as  $\text{pl}(\text{Hot} \mid \text{data})$  is also a plausibility function. The value of  $\text{pl}$  on non elementary elements of its domain is not directly available, even though it can be derived from the GBT results. When  $\Theta$  generalizes the temperature's domain in order to have a non-binary variable, one has:

$$\text{pl}(A \mid \text{data}) = \text{pl}(\text{data} \mid A) = 1 - \prod_{\theta_i \in A} (1 - \text{pl}(\text{data} \mid \theta_i)) \quad \text{for all } A \subseteq \Theta.$$

*Note:* In probability theory, the likelihood  $l(A \mid \text{data})$  for  $|A| > 1$  is sometimes defined as:

$$l(A \mid \text{data}) = \max_{\theta_i \in A} l(\theta_i \mid \text{data}).$$

In that case the likelihood function is a possibility function (Zadeh, 1978, Smets, 1982) as it satisfies:

$$l(A \cup B \mid \text{data}) = \max(l(A \mid \text{data}), l(B \mid \text{data})).$$

These coefficients  $\frac{\text{pl}(\text{data} \mid \text{Hot})}{\text{pl}(\text{data} \mid \text{Cold})}$  and  $\frac{l(\text{Hot} \mid \text{data})}{l(\text{Cold} \mid \text{data})}$  by which the prior odds are multiplied to obtain the posterior odds are presented in table 4.1.8. They are similar in the first and third case. Discrepancies appear in case two, where the odds are larger than one in the TBM analysis, indicating that Your belief that the temperature is Hot is increased by the data, whereas in the probability approach, the odds is smaller than one, indicating that Your beliefs that the temperature is Hot is decreased by the data. Suppose You must take a decision according to the

variation of Your beliefs, like to call John Doe if beliefs that Hot increases, and James Ickx if it decreases. What would You do? In the TBM approach, You would call John and in the probability approach, James. So choosing the theory is not just a minor academic issue.

Case	1	2	3
Odds Ratio for Hot	Coefficient by which prior odds are multiplied		
TBM	0.02	1.44	100.00
Probability Th.	0.00	0.81	199.00

**Table 4.1.8:** For the three sets of observations of table 4.1.5, values of the coefficients by which the prior odds for the temperature status ‘hot’ is multiplied to compute the posterior odds.

The results depend on the .5 value used for the missing probability  $P(\text{Red} | \text{broken})$ . As already mentioned, using other value does not help as similar examples can always be built. A more cautious approach sometimes proposed is just to accept that the missing probability can take any value between 0 and 1, and to perform a sensitivity analysis. Unfortunately results are useless. Given the data (and in fact given any observed data) the factor by which the prior odds are multiplied can taken any value between 0 and  $\infty$ , what is already known before looking at the sensors (by definition, the factor is always between 0 and  $\infty$ ). Using meta-probability, i.e., a probability measure over the value of the unknown probability  $P(\text{Red} | \text{broken})$  might be thought off, but would require the assessment of these meta-probabilities (remembering that some authors even contest that meta-probabilities exists!)

What is the best model? This question cannot be answered, unfortunately, because its answer requires a definition of ‘a good model’. The advantage of the TBM comes from the fact it takes into account only the information really available, whereas too often the Bayesian is forced into filling missing probabilities by pure guesses, what might be called an artificial hyper-probabilization. In a state of total ignorance, as held by the technician using the sensors, no value for the missing probability can be justified (except the 0.5 that results from the Principle of Insufficient Reason, a highly dangerous principle that leads to many contradictions, and is usually denied by the Bayesians themselves). The introduction of a maximum entropy principle or of a meta-probability does not really solve the problem, it just displaces the problem. The only way to select the ‘good model’ is obtained by a close examination of the assumptions underlying the two competing models, and then deciding which set of assumptions is the most appropriate for the problem to be solved (Smets, 1993b).

**4.2. Finding the broken circuit.**

Suppose an equipment that is made out of 15 circuits denoted  $c_i$ :  $i = 1, 2, \dots, 15$ . Each circuit is necessary so that the equipment can work adequately. It happens that the equipment is broken. So You know that at least one circuit is broken. You assume that only one circuit is broken. The reason for such an assumption resides in Your opinion that two circuits being simultaneously broken is not plausible. Generalization by relaxing this assumption is possible, but the assumption greatly simplifies the analyses.

The diagnostic problem consists in finding which circuit  $c_i$  is broken. You can use five sensors  $S_j$ :  $j = 1, 2, \dots, 5$  to check the working condition of the equipment. The sensor's outcome  $s_j$  is binary and we use the convention:

- $s_j = 0$  if the sensor  $S_j$  does not detect a failure,
- $s_j = 1$  if the sensor  $S_j$  detects a failure.

You know the relation between the circuits' status and the sensors' outcome. Let  $a_{ij}$  be an indicator that expresses what You know about the relation between circuit  $c_i$  status and sensors  $S_j$ 's outcome, with the convention

- $a_{ij} = 1$  :  $s_j$  depends on  $c_i$ , in which case if  $c_i$  is broken then  $s_j = 1$
- $a_{ij} = 0$  :  $s_j$  does not depend of  $c_i$ .
- $a_{ij} = ?$  : You don't know the dependency between  $s_j$  and  $c_i$ .

The last value for  $a_{ij}$  reflects that You might not know all the details about the structure of the failed equipment and You just don't know if  $S_j$ 's outcome will be influenced by  $c_i$ 's status.

Table 4.2.1 presents the value of  $a_{ij}$  for the 15 circuits  $c_i$  and the 5 sensors  $S_j$ .

circuit	sensor					S = 1	S = 0	S = 1
	1	2	3	4	5	$M_1$	$M_2$	$M_3$
1	1	0	?	1	1	1	1	1
2	?	1	0	0	1	1		
3	1	0	1	1	1	1	1	1
4	0	0	0	?	0		1	
5	0	?	0	0	?		1	
6	0	1	1	1	1			1
7	1	1	?	1	?	1		1
8	?	1	0	1	1	1		
9	1	?	0	0	1	1	1	
10	1	0	?	1	1	1	1	1
11	1	0	1	1	1	1	1	1
12	0	0	1	1	0		1	1
13	?	?	?	1	1	1	1	1
14	1	?	0	0	1	1	1	
15	0	?	1	?	1		1	1

**Table 4.2.1.** Value of  $a_{ij}$  for the 15 circuits  $c_i$  and the 5 sensors  $S_j$ . Right part presents the sets  $M_i$  obtained when  $s_1 = 1$ ,  $s_2 = 0$  and  $s_3 = 1$ .

Let  $A_j = \{c_i: a_{ij} = 1\}$ ,  $B_j = \{c_i: a_{ij} = 0\}$  and  $C_j = \{c_i: a_{ij} = ?\}$ .

So for instance You know that:

- whenever one of the circuit in  $A_1$  is broken then  $s_1 = 1$ ,
- if none of them is broken then  $s_1$  might be 0
- $s_1$  is unrelated to the status of circuits in  $B_1$ , so if a circuit in  $B_1$  is broken then  $s_1 = 0$ , and You don't know if the status of the circuits in  $C_1$  influence or not the outcome of  $S_1$ .



Unfortunately, the sensors themselves can be broken and You only know their reliability. You consider that sensor  $S_1$ 's reliability is 0.95. It means that in Your opinion, there is a probability 0.95 that the sensor  $S_1$  is in working condition and a probability 0.05 that sensor  $S_1$  is not in a working condition.

- In the first case, sensor  $S_1$  outcome is meaningful, in which case and by pure logic, if  $s_1 = 0$  you know that those circuits in  $A_1$  are all in working condition, so the broken circuit belongs to  $B_1 \cup C_1$ , and if  $s_1 = 1$  You know that the broken circuit is one of those in  $A_1 \cup C_1$ , and none of  $B_1$ .

- In the second case, You can as well forget the outcome of sensor  $S_1$  as its outcome is meaningless when it comes to decide which is the broken circuit.

Table 4.2.2 presents the reliabilities of the 5 sensors.

Sensor				
1	2	3	4	5
0.95	0.85	0.80	0.75	0.70

**Table 4.2.2:** Reliabilities of the 5 sensors.

The sensors  $S_1$ ,  $S_2$  and  $S_3$  have been applied and their respective outcomes were 1, 0 and 1 (see table 4.2.1). In the columns  $M_1$  to  $M_3$ , the value 1 indicates which of the 15 circuits might be broken if the sensor is reliable. These three sets are denoted  $M_j$ ,  $j=1, 2, 3$ . Given the sensors' outcomes,  $M_1 = A_1 \cup C_1$ ,  $M_2 = B_2 \cup C_2$  and  $M_3 = A_3 \cup C_3$ . Indeed if sensor  $S_1$  is reliable (probability 0.95) then the broken circuit may not be one in  $B_1$ , and it can be any in  $A_1$  or  $C_1$  (as maybe those circuits in  $C_1$  might influence  $S_1$ 's outcome). Identically if sensor  $S_2$  is reliable (probability 0.85), then the broken circuit may not be one of  $A_2$  (as in that case  $s_2$  would be 1, contrary to the observation  $s_2 = 0$ ) and might be any of  $B_2$  or  $C_2$ , etc...

What we obtain for the three sensors' outcomes are just three simple support functions  $m_j$ :  $j=1, 2, 3$ , on  $\Omega = \{c_i: i= 1, \dots, 15\}$  with  $\Omega$  and  $M_j$  being their focal elements. So:

- 1)  $m_1(M_1) = .95$  and  $m_1(\Omega) = .05$ ,
- 2)  $m_2(M_2) = .85$  and  $m_2(\Omega) = .15$ , and
- 3)  $m_3(M_3) = .80$  and  $m_3(\Omega) = .20$ .

Ω circuit	bba on set of circuits Ω								circuit c <sub>i</sub> is broken plausib    BetP	
	0.65	0.16	0.11	0.03	0.03	0.01	0.01	0.00		
	M <sub>1</sub> M <sub>2</sub> M <sub>3</sub>	M <sub>1</sub> M <sub>2</sub> Ω	M <sub>1</sub> Ω M <sub>3</sub>	M <sub>1</sub> Ω Ω	Ω M <sub>2</sub> M <sub>3</sub>	Ω M <sub>2</sub> Ω	Ω Ω M <sub>3</sub>	Ω Ω Ω		
1	1	1	1	1	1	1	1	1	1.00	0.18
2				1				1	0.03	0.00
3	1	1	1	1	1	1	1	1	1.00	0.18
4						1		1	0.01	0.00
5						1		1	0.01	0.00
6							1	1	0.01	0.00
7			1	1			1	1	0.15	0.02
8				1				1	0.03	0.00
9		1		1		1		1	0.20	0.03
10	1	1	1	1	1	1	1	1	1.00	0.18
11	1	1	1	1	1	1	1	1	1.00	0.18
12					1	1	1	1	0.05	0.01
13	1	1	1	1	1	1	1	1	1.00	0.18
14		1		1		1		1	0.20	0.03
15					1	1	1	1	0.05	0.01

**Table 4.2.3:** Values of the bba  $m_{1 \wedge 2 \wedge 3}$ , with their focal element, obtained by combining conjunctively the three simple support functions induced by the three observations.

The conjunctive combination of these three simple support functions results in the bba given in table 4.2.3. We have  $m_{1 \wedge 2 \wedge 3}(M_1 \cap M_2 \cap M_3) = .95 * .85 * .80 = .65$ , and  $m_{1 \wedge 2 \wedge 3}(M_2 \cup M_3) = .05 * .85 * .80$ , etc... The next column of table 4.2.3 lists the values of the plausibility computed from  $m_{1 \wedge 2 \wedge 3}$  on the singletons of  $\Omega$ . Low plausibilities as for circuits  $c_2, c_4, c_5, c_6, c_8, c_{12}, c_{15}$  indicate that the broken circuit is none of them. Table 4.2.3 then lists the value of the pignistic probability  $BetP$  given to each singleton and computed from  $m_{1 \wedge 2 \wedge 3}$ . It indicates that the broken circuit is probably one of  $c_1, c_3, c_{10}, c_{11}$  or  $c_{13}$ , i.e., those circuits in  $M_1 \cap M_2 \cap M_3$ , and pointed as broken by the three sensors when they are reliable.

Given that level of beliefs about the broken circuit, You must now decide if You are going to use sensor  $S_4$  or sensor  $S_5$  as a next step in Your diagnostic procedure. Table 4.2.1 presents the value of  $a_{ij}$  for sensors  $S_4$  and  $S_5$ , and table 4.2.2 gives their reliability. The procedure consists then in deciding which one of the two sensors is the most efficient for establishing the diagnosis when only one of them is used. Let us accept that You want to minimize the entropy computed from the pignistic probability. This means that You accept that this entropy is an appropriate measure for the information available to You when decision must be made (this is a quite common attitude in classical probability, but other attitude could be adopted as well). The entropy related to the pignistic probability  $BetP$  is given by:

$$Entr(BetP) = \sum_{c_i: c_i \in \Omega} BetP(c_i) \cdot \log(BetP(c_i))$$

With  $BetP$  computed from  $m_{1 \wedge 2 \wedge 3}$ , the value of the entropy is 1.94.

To compute the benefit You could expected from using the sensors  $S_4$ , You suppose first that  $s_4 = 0$ , You compute the simple support function  $m_4$  on  $\Omega$ , You combine it conjunctively with  $m_{1\wedge 2\wedge 3}$ , You obtain the bba  $m_{1\wedge 2\wedge 3\wedge 4}$ , You compute the pignistic probability  $BetP$  on  $\Omega$  induced by  $m_{1\wedge 2\wedge 3\wedge 4}$  and then the entropy of  $BetP$ . You perform similar computation with  $s_4 = 1$  and obtain another entropy. The two entropies so obtained are denoted  $Entr_{4=0}$  and  $Entr_{4=1}$ .

You must then compute from  $m_{1\wedge 2\wedge 3}$  Your belief  $bel_s$  that  $S_4$ 's outcome will be 0 or 1, respectively. This belief is obtained from Your belief  $bel_{1\wedge 2\wedge 3}$  as following: given  $a_{i4}$  You know that  $s_4 = 0$  whenever the broken circuit belongs to  $A_4$ , and that  $s_4 = 1$  whenever the broken circuit belongs to  $B_4$ . So  $pl_s(s_4 = 0) = pl_{1\wedge 2\wedge 3}(\text{broken circuit} \in A_4)$  and  $pl_s(s_4 = 1) = pl_{1\wedge 2\wedge 3}(\text{broken circuit} \in B_4)$ . This plausibility function  $pl_s$  is transformed into pignistic probabilities  $BetP$  on the betting frame  $\{s_4 = 0, s_4 = 1\}$ . Then in order to compute the entropy you might expect by testing the equipment with  $S_4$ , you compute:

$$Entr_4 = BetP(s_4 = 0) \cdot Entr_{4=0} + BetP(s_4 = 1) \cdot Entr_{4=1}.$$

Identical computation is performed with  $S_5$ . Results are presented in table 4.2.4. They show that sensor  $S_4$  is the best one to apply, as it minimizes the expected entropy.

	Outcome	Entropy	BetP	E(Entr)
Sensor 4	$s_4 = 0$	2.19	0.12	1.86
	$s_4 = 1$	1.82	0.88	
Sensor 5	$s_5 = 0$	2.02	0.10	1.93
	$s_5 = 1$	1.92	0.90	

**Table 4.2.4:** Entropies  $E(Entr)$  one could expected by applying sensors 4 and 5 individually.

You can still consider other actions in that example. Instead of applying new sensors, You might consider finding the exact value of  $a_{ij}$ ,  $j = 1$  to 3 for the case where  $a_{ij} = ?$ . This could be achieved by calling the factory and checking the original plans (that You do not have). You consider each case where  $a_{ij} = ?$ . Suppose  $a_{kl} = ?$ , You assume that  $a_{kl} = 1$  and compute the entropy  $Entr_1$  computed from the pignistic probability You would obtained by putting  $a_{kl} = 1$  in table 4.2.1. You repeat the procedure by assuming that  $a_{kl} = 0$ , what leads to  $Entr_0$ . Suppose Your a priori belief about the value of  $a_{kl}$  is vacuous, then the entropy You could expect by learning the value of  $a_{kl}$  is then:

$$E_{kl}(Entr) = .5 Entr_0 + .5 Entr_1.$$

Table 4.2.5 presents the value of the expected entropy for every case where  $a_{ij} = ?$  for  $j = 1, 2, 3$ . It indicates that the richest information would be obtained by learning the value of  $a_{13,1}$  i.e. the link between circuit  $c_{13}$  and sensor  $S_1$ .

Nevertheless applying sensor  $S_4$  is still better from a minimal entropy point of view. So if there is no financial difference between applying a new sensor or getting the value of  $a_{13,1}$  Your best decision would be to apply sensor  $S_4$ .

Circuit	Sensor	$a_{ij}$	Entropy	BetP	E(Entr)
2	1	0	1.93	0.5	1.94
		1	1.94	0.5	
8	1	0	1.93	0.5	1.94
		1	1.94	0.5	
13	1	0	1.80	0.5	1.87
		1	1.94	0.5	
5	2	0	1.94	0.5	1.94
		1	1.94	0.5	
9	2	0	1.94	0.5	1.92
		1	1.89	0.5	
13	2	0	1.94	0.5	1.89
		1	1.84	0.5	
14	2	0	1.94	0.5	1.92
		1	1.89	0.5	
15	2	0	1.94	0.5	1.93
		1	1.92	0.5	
1	3	0	1.84	0.5	1.89
		1	1.94	0.5	
7	3	0	1.89	0.5	1.92
		1	1.94	0.5	
10	3	0	1.84	0.5	1.89
		1	1.94	0.5	
13	3	0	1.84	0.5	1.89
		1	1.94	0.5	

**Table 4.2.5:** Computation of the entropies one could expect by resolving the ambiguities left for the relation between the sensors and the circuits (those (i,j) with  $a_{ij} = ?$  in table 4.2.1)

Generalization when cost and utilities are involved or when a priori beliefs exists is of course possible. It will not change the philosophy of the present approach..

### 4.3. Lung diseases.

We analyze a highly simplified medical diagnosis problem in order to illustrate the use of the Generalized Bayesian Theorem (GBT). It also shows how beliefs can be assessed. Let  $\Theta = \{\text{tuberculosis, lung cancer, bronchitis, none}\}$ , where the categories are considered as mutually exclusive and where ‘none’ represents the absence of the three real disease categories. Two symptoms can be collected: the lung X-ray that can be positive or negative, and the fever that can be none, mild or severe.

We assume that one and only one disease can be present at the same time (this hypothesis could be relaxed but the complexity of the computation should increase seriously and the example would loose its illustrative capacity).

X-ray	tuberculosis	lung cancer	bronchitis	none
$\emptyset$	0	0	0	0
Positive	0.8	0.4	0.1	0
Negative	0	0	0.7	0.9
$\Omega$	0.2	0.6	0.2	0.1

BetP	tuberculosis	lung cancer	bronchitis	none
Positive	0.9	0.7	0.2	0.05
Negative	0.1	0.3	0.8	0.95

**Table 4.3.1:** Conditional beliefs over the X-ray symptoms within the four categories. The upper part presents the four bba, and the lower part presents the pignistic probabilities BetP within each categories.

Your knowledge about the relation between the diseases and the symptoms is represented by Your conditional beliefs about the symptoms that would be observed if You knew that the patient belongs to the disease category  $\theta \in \Theta$ . This is available for each diseases category  $\theta \in \Theta$ . Table 4.3.1 presents the bba related to the conditional belief function you should have about the value of the X-ray outcomes. You assume first that  $m(\emptyset) = 0$  as You feel there is no contradiction in your conditional beliefs. Their constructions are based on the pignistic probabilities You would have if You had to bet on which symptom will be present if the patient belongs to category  $\theta$ . The bottom part of table 4.3.1 presents these probabilities BetP. When patient presents a tuberculosis, You would bet with 0.9 that the X-ray is positive, and 0.1 that is negative. Many bba could be built on the X-ray domain and that are compatible with the pignistic probability just defined. But it happens that You feel that when tuberculosis is present, You have no reason to belief that the X-ray might be negative. Hence You impose that  $m(\text{Negative}) = 0$ . In that case there is only one bba that fits with BetP and  $m(\text{Negative}) = 0$ , the one given in table 4.3.1 that allocates  $m(\text{Positive}) = 0.8$  and  $m(\Omega) = 0.2$  (where  $\Omega$  denotes systematically the domain of the variable under consideration, the X-ray here).

Data for the lung cancer are obtained identically, and those for none also except the null mass is forced for Positive.

For the bronchitis, the assessment of bba is a little more subtle. You don't want to assume the same constraint as in the previous cases. On the betting frame  $\mathcal{B}_1 = \{\text{Positive}, \text{Negative}\}$  Your BetP<sub>1</sub> are 0.2 and 0.8, respectively (see table 4.3.1). Let the bba on  $\mathcal{B}_1$  that must be assessed be denoted by  $m_1$ . Now we build a second betting frame  $\mathcal{B}_2$  based on an adapted domain for the X-ray outcome. Suppose there are two forms of positive X-ray according to the size of the lesion: Small or Large. So the second betting frame  $\mathcal{B}_2 = \{\text{X-ray Positive with a Small lesion}, \text{X-ray Positive with a Large lesion and, X-ray Negative}\}$ . If You had to bet on the lesion size, You would be totally ignorant about it. So in Your opinion the bba  $m_2$  on the set  $\mathcal{B}_2$  should give no support to Positive Small or to Positive Large, what means that the bba  $m_2$  on  $\mathcal{B}_2$  satisfies:

$$m_2(\text{Positive Small}) = m_2(\text{Positive Large}) = 0$$

$$m_2(\text{Positive}) = m_1(\text{Positive}).$$

$$m_2(\text{Negative or Positive Small}) = m_2(\text{Negative or Positive Large}) = 0$$

$$\begin{aligned}
m_2(\text{Negative}) &= m_1(\text{Negative}). \\
m_2(\Omega) &= m_1(\Omega) \\
m_2(\emptyset) &= m_1(\emptyset) = 0.
\end{aligned}$$

Suppose You are asked to provide Your pignistic probabilities on  $\mathcal{B}_2$  and that they are:

$$\text{BetP}_2(\text{Positive Small}) = \text{BetP}_2(\text{Positive Large}) = 0.117$$

and  $\text{BetP}_2(\text{Negative}) = 0.767$ .

So we have to solve the following set of equations:

$$\text{BetP}_1(\text{Positive}) = 0.2 = m_1(\text{Positive}) + m_1(\Omega)/2$$

$$\text{BetP}_1(\text{Negative}) = 0.8 = m_1(\text{Negative}) + m_1(\Omega)/2$$

$$\text{BetP}_2(\text{Positive Small}) = 0.117 = m_1(\text{Positive})/2 + m_1(\Omega)/3$$

$$\text{BetP}_2(\text{Positive Large}) = 0.117 = m_1(\text{Positive})/2 + m_1(\Omega)/3$$

$$\text{BetP}_2(\text{Negative}) = 0.767 = m_1(\text{Negative}) + m_1(\Omega)/3$$

$$m_1(\text{Positive}) + m_1(\text{negative}) + m_1(\Omega) = 1.$$

The only solution is  $m_1(\text{Positive}) = 0.1$ ,  $m_1(\text{Negative}) = 0.7$  and  $m_1(\Omega) = 0.2$  (see table 4.3.1)

Similar assessments are performed for the fever symptom. Given the betting probabilities obtained within each of the four disease categories, You build the least committed belief function over the fever status which pignistic transformation would lead back to the pignistic probability  $\text{BetP}$  observed. The results are given on table 4.3.2.

fever	tuberculosis	lung cancer	bronchitis	none
$\emptyset$	0	0	0	0
none	0	0.45	0	0.91
mild	0	0	0	0
none-mild	0	0.4	0	0.06
severe	0.1	0	0.35	0
none-sev	0	0	0	0
mild-sev	0.6	0	0.5	0
$\Omega$	0.3	0.15	0.15	0.03

BetP	tuberculosis	lung cancer	bronchitis	none
none	0.1	0.7	0.05	0.95
mild	0.4	0.25	0.3	0.04
severe	0.5	0.05	0.65	0.01

**Table 4.3.2:** Conditional beliefs over the fever symptoms within the four categories. The upper part presents the four bba, and the lower part presents the pignistic probabilities  $\text{BetP}$  within each categories.

Three methods for belief assessments are thus illustrated. The first is based on the assumption of special constraints, the second on varying betting schemes, and the third on the least commitment principle ('never allocate more belief than necessary').

In order to apply the GBT, we need the belief over the symptom space within each disease category. This is obtained by accepting the idea that the symptoms are cognitively independent (the concept that generalizes the idea of stochastic independence). It means that the plausibility to observed a Positive X-ray and a mild fever in disease category  $\theta$  is the product of the plausibility to observe a Positive X-ray in  $\theta$  and of observing a mild fever in  $\theta$ . The analogy with the probabilistic case is obvious. Similar computation is done for every combination of symptoms within each disease category.

Knowing the belief over the symptoms within each diseases category, it is just a matter of applying the GBT in order to produce the belief over the disease classes given the observed symptoms.

Suppose then that the X-ray is positive and the clinician only knows that the patient has ‘mild or severe’ fever. The exact value of the fever status happens to be unknown to the clinician (the patient speaks of fever, but don’t provide any detail about it). All that is needed in that case are the conditional of Positive X-ray and mild-severe fever. They are:

$$pl(\text{Positive \& mild-severe}|\theta) = pl(\text{Positive}|\theta) pl(\text{mild-severe}|\theta).$$

Table 4.3.3 presents the details of the computation.

Symptom	Status	tuberculosis	lung cancer	bronchitis	none
X-ray	Positive	1.000	1.000	0.300	0.100
Fever	mild-sev.	1.000	0.550	1.000	0.090
Combin.	&	1.000	0.550	0.300	0.009

**Table 4.3.3:** Values of the conditional plausibility needed for the computation of the beliefs given X-ray Positive and fever mild-severe.

Table 4.3.4 presents the values of the belief function  $bel$ , the plausibility function  $pl$  and the pignistic probability  $BetP$  for each disease category individually. They are obtained by the application of the GBT. In that case, the most probable diagnosis is tuberculosis ( $BetP = 0.628$ ) and there is no conflict in the induced belief function over the disease categories ( $conflict = 0$ ). Notice that the plausibility of the disease categories given the observed symptoms are equal to the plausibility of the observed symptoms given the disease category, what is indeed always true and reflects the equality:

$$pl(A|B) = pl(A\&B) = pl(B|A) \text{ for any } A, B \subseteq \Omega.$$

X-ray	Fever	Conflict		tuberculosis	lung cancer	bronchitis	none
positive	mild-sev	0.000	bel	0.312	0.000	0.000	0.000
			pl	1.000	0.550	0.300	0.009
			BetP	0.628	0.247	0.122	0.003
negative	mild	0.207	bel	0.046	0.102	0.292	0.021
			pl	0.180	0.330	0.585	0.090
			BetP	0.133	0.262	0.542	0.064
positive	none-mild	0.000	bel	0.000	0.072	0.000	0.000
			pl	0.900	1.000	0.195	0.100
			BetP	0.407	0.493	0.066	0.033
positive	$\Omega$	0.000	bel	0.000	0.000	0.000	0.000
			pl	1.000	1.000	0.300	0.100
			BetP	0.436	0.436	0.098	0.031
$\Omega$	mild-sev.	0.000	bel	0.000	0.000	0.000	0.000
			pl	1.000	0.550	1.000	0.090
			BetP	0.397	0.179	0.397	0.026

**Table 4.3.4:** Beliefs induced by the GBT on the four categories. Five sets of symptoms are considered. For each of them, the table presents the values of bel, pl and BetP for each category.  $\Omega$  denotes the domain of the symptoms. Notations like mild-sev. means mild or severe,...

Suppose now that the patient has a negative X-ray and a mild fever. The pignistic probabilities indicate that the most probable category is bronchitis (BetP = 0.542). In this case, the conflict is 0.207, a medium value that results from the fact that the patient has a negative X-ray, what is compatible with bronchitis and 'none', but he has a mild fever, what is not exactly compatible with bronchitis nor with 'none'. The 0.207 conflict enhances this weak contradiction.

If the patient has a positive X-ray and does not suffer from severe fever, then the most probable diagnosis would be lung cancer (bet P = 0.493). This example just illustrates how easy it is to compute the requested data.

The interest of the method resides in the fact that:

- 1) there is a measure of conflict. It indicates the presence of some incoherence between the symptoms.
- 2) the prior belief over the disease is vacuous. The method developed does not require any specific a priori belief about the disease categories, a major advantage of the method. If we had some a priori belief about the disease, this belief would have been conjunctively combined with the belief function induced by the symptoms over the disease categories in the presence of a vacuous a priori belief over the disease categories.

Finally, to show that the GBT behaves appropriately when symptoms are combined, suppose successively that we only know that the patient has a positive X-ray, and then that the fever is mild or severe (see table 4.3.4 last two cases). Table 4.3.5 presents the values of the bba (the missing values are null) in both cases and their combination. The result of this combination is nothing but the data that were summarized in table 4.3.4, case 1.



non null bba	X-ray positive	Fever mild-severe	Combination
tub			0.312
tub-lcanc	0.630		0.382
tub-bronch		0.410	0.134
tub-lcanc-bronch	0.270	0.501	0.164
tub-none			0.003
tub-lcanc-none	0.070		0.003
tub-bronch-none		0.041	0.001
$\Omega$	0.030	0.050	0.001

**Table 4.3.5:** Non null values of the bba induced by the GBT on the four categories given the observed X-ray symptom and given the observed fever symptom. Their combination by Dempster's rule of combination results in the bba presented in table 4.3.3 for the first symptoms combination. The data for the individual symptoms are those consider in table 4.3.3 as cases 4 and 5.

#### 4.4. Radio-active leakage location.

Suppose You discovers that there is some abnormal radio-activity is a given region, and You suspect it results from a leakage in one of the 7 nuclear waste disposal sites in that region. You know the location of the disposal sites. There are 21 locations where radio-activity measurements can be obtained. For each disposal site and for each measurement location, You provide Your conditional belief about what would be the outcome of the measurement (e.g., positive or negative) given the radio-active material is leaking at that disposal site. You know the cost of collecting each radio-activity measurement. You also know, for each disposal site, the cost of exploring it and the consequences of a delay in taking appropriate actions if the leakage happens to be at that site. Details are given in Xu et al. (1993, 1996).

Given the observed radio-activity measurement obtained from a selected sample of sites, You assess Your beliefs about the location of the leakage and the optimal action. The computation can hardly be performed on a piece of paper as for the previous examples, but the evaluation of the beliefs and the optimal decision becomes easily manageable when using the Evidential Networks software (Xu, 1992a) and the Decision Support System (Xu, 1992b), both based on the Valuation-Based Systems (Shafer et al., 1987, Shenoy and Shafer, 1990, Shenoy, 1992). The architecture of the integrated software is given in Xu (1995).

With the same information, You can also precompile the optimal strategy to follow if some abnormal radio-activity is detected, just as we did in the broken circuit example (section 4.2). The precompile strategy will tell its users which sample to collect, which action to taken given the observation, like 'proceed with sampling at that site', or 'explore that disposal site'... Such precompile strategy could then be introduced in the adequate safety procedures.

Up to here, all we have done mimic the Bayesian approach, both in the diagnostic process as in the precompilation of an optimal strategy. The only difference is that the relations between the variables are represented by belief functions instead of probability functions. Otherwise utilities

and expected utilities are similarly computed (with the pignistic probabilities in the TBM). The advantage of the TBM is that one does not have to feed into the model those many probabilities required by a full flesh probabilistic approach, what often goes far into the thousands, even though most of them are totally unsupported and reflects pure uneducated and unjustified guesses. The TBM uses only what is known, not what is invented (usually by a blind application of the Principle of Insufficient Reason). This reflects the ability of the TBM to represent any level of beliefs, up to total ignorance, a task hardly achievable within probability theory.

Nevertheless there is one adaptation of the scenario where the TBM really beats the most classical probability approach. Suppose that You are not aware that the Army might have created a disposal site in the region You are exploring. Military Secrets are sometimes so strong that You are ignorant about the existence of such a Secret Site, and of course, You have no idea whatsoever where it might be located, and how the radio-activity measurements can be influenced by a leakage happening in that Secret Site. With the TBM such a case can easily be solved by introducing the extra Secret Site  $\theta_\omega$ , and assuming vacuous belief functions on the radio-activity measurements outcomes when a leakage occurs in the Secret Site. All these adaptations are straightforward and do not need any modification of the model. Within the Bayesian approach, I prefer to leave such a problem to a Bayesian specialist, but in my personal opinion, there is not real way to solve realistically such a problem without introducing totally artificial assumptions.

## **5. Belief networks and evidential networks.**

In the radio-activity leakage, we mentioned the use of evidential networks to compute the required beliefs. In probability approach, similar computations are achieved with the Bayesian networks (Pearl, 1988, Lauritzen and Spiegelhalter, 1988). Even if the two approaches share similarities, they also have major conceptual differences that are often neglected. So we detail these hereafter.

In both cases, one starts with a set  $U$  of variables  $X_i$ ;  $i=1, 2, \dots, n$ , each with its finite domain  $D_i$ . In the probability approach one assumes the existence of an joint probability function over the product space  $X = X_1 \times X_2 \times \dots \times X_n$ , whereas the TBM approach constructs a belief function over the space  $X$ . In the probability approach, one assumes the existence of a very large number of independence constraints, thanks to which the joint probability function on  $X$  can be represented by a few appropriate conditional probability functions between subsets of variables of  $U$ . Links between variables indicates their relation, the absence of a link indicating that the two variables are conditionally independent given the other variables. With the TBM, one does not assume the existence of a belief function on  $U$  that could be represented by a small number of belief functions defined on smaller subsets of  $U$ . Each piece of information induced a belief function on a subset of  $U$ , and the joint belief function on  $X$  is built from these belief functions. The links indicate the belief functions constraints between those variables connected by the link. The absence of a link between a subset  $A$  of variable in  $U$  reflects that the belief function

between those variables in A is vacuous. In the TBM, each time a new piece of evidence is collected, one just adds the induced belief function to the network. This addition does not change what was previously introduced in the network, it just increases the information represented by it. In probability theory, adding a new probability function to the network consists either in changing preexisting probabilities or changing one's mind about a previously assumed independence. Knowing more, as in the TBM, is different from shifting from an independence assumption toward a dependency. In the TBM, You just know more. Going from a vacuous belief function toward a belief function is what occurs when You learn a new piece of evidence. In the probability approach, You change Your mind about a previously assumed property, what means You acknowledge Your independence assumption was inappropriate.

The belief functions can be joint belief functions as in the original network as presented by Shenoy and Shafer (1990), but they can as well be conditional belief functions (Xu and Smets, 1995). The simplest (if maybe not most efficient way to cope with conditional belief functions consists in creating their ballooning extension on the product space (Smets, 1993a), and then combining conjunctively those joint belief functions so derived, in which case we are back to the original network. So the evidential network can cope with both joint and conditional belief functions, and the critics that claim that evidential networks cannot cope with conditional information are just wrong.

Besides there is another serious discrepancy between the Bayesian networks and the evidential networks. In the TBM, You can perfectly introduce two belief functions between variables  $X_1$  and  $X_2$ . It would mean that You have two sources of information, and the two induced belief functions are combined into a single belief function on  $X_1 \times X_2$ . In the Bayesian networks, You can freely assign the conditional probability function on  $X_1$  given  $X_2$ . Once this assignement is done, You may not introduce another conditional probability function on  $X_1$  given  $X_2$ . You could introduce a conditional probability function on  $X_2$  given  $X_1$ , but this new probability function must be built very carefully if You require, what is compulsory, that the two conditional probability functions are compatible with a unique underlying joint probability function on the product space. Requiring that Bayesian networks are directed acyclic graphs consists in avoiding such dangerous construction. In the evidential networks, this is not required, that networks are directed acyclic or not is irrelevant, they all finished as undirected graphs thanks to the ballooning extension.

## **6. Conclusions.**

We have summarized the possibilities of the transferable belief model in domains related to the diagnostic process, and shown its strengths within some applications. The interest and capacities of the TBM do not stop so short of course. Many problems of probability theory have already be generalized in order to be handled by the TBM. In particular, the TBM can be used for cluster analysis (Schubert, 1995), for discriminant analysis (Denoeux, 1995), for expert opinions pooling (Smets, 1992a), for plausible reasoning (Benferhat et al., 1995), for

auditing procedures (Srivastava, 1995, Van den Acker and Vanthienen, 1996), for classification (Lohmann, 1991), for decision making (Xu et al., 1996) etc... In particular, with the TBM, it is possible to perform a discriminant analysis even when the classes of the cases in the learning sets are not exactly known, but only known to belong to a set of possible classes. Future use of the TBM in real worlds applications will permit to assess its value.

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