Belief Functions on Real Numbers.

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Abstract

We generalize the TBM (transferable belief model) to the case where the frame of discernment is the extended set of real numbers $\mathcal{R} = [-\infty, \infty]$, under the assumptions that 'masses' can only be given to intervals. Masses become densities, belief functions, plausibility functions and commonality functions become integrals of these densities and pignistic probabilities become pignistic densities. The mathematics of belief functions become essentially the mathematics of probability density functions on $\mathcal{R}^2$.

1 Introduction

In this paper, we accept that beliefs are quantified by belief functions, as described in the transferable belief model (TBM) (Smets & Kennes, 1994). Classically belief functions are defined on finite frames of discernment. We present some extensions of the belief function theory to $\mathcal{R} = [-\infty, \infty]$, the extended set of real numbers. We will consider the case where the focal elements are closed intervals in $\mathcal{R}$. We will work essentially on the frames of discernment $[0, 1]$ and $\mathcal{R}$, but we will also provide some hints for belief functions on $\mathcal{R}^n$.

Classical material about belief functions and the transferable belief model can be found in (Shafer, 1976; Smets & Kennes, 1994; Smets, 1998).

We consider successively belief functions on $[\alpha, \beta], -\infty < \alpha < \beta < \infty$, just to explain the origin of the generalization to non countable domains, and belief functions on $\mathcal{R}$, presenting the generalization of most of the concepts and relations encountered in the finite case.

New concepts are introduced, among which those of credal variables, of characteristic functions and of least committed isopignistic belief functions induced by the knowledge of their pignistic transformations.

Belief functions on $\mathcal{R}$, in particular figures 1 and 2 and relations (1) to (12), were already presented in (Smets, 1978). Figure 2 and relations (1) to (3) were also published in (Strat, 1984). An application of belief functions on $\mathcal{R}$ is presented in (Ristic & Smets, 2004)

Shafer's thesis (Shafer, 1974) and (Shafer, 1979) also discusses generalization of belief function theory to more abstract spaces. Shafer requires a 'condensability' property, i.e. $pl(A) = \sup\{pl(B) : B \subseteq A, B \text{ is finite}\}$. This constraint is not required by the belief functions we describe in this paper.

In (Kohlas & Monney, 1995, chapter 16), authors present the relations (1) to (12) in a much more formal way than done here, some having already been considered in (Dempster, 1968). Belief functions on $\mathcal{R}$ are also mentioned in (Dempster, 1990; Almond, 1995). Belief functions on $\mathcal{R}$ are used in assumption-based statistical inference (Monney, 2003; Kohlas & Monney, 2004). Generalization of belief functions on some semi-lattices are considered in (Kohlas, 2003b, 2003a).

In (Liu & Shenoy, 1995; Liu, 1996; Liu, Shenoy, & Shenoy, 2003), the authors introduce their so-called linear belief functions. They consider belief functions in the framework defined by Dempster. They assume a continuous probability density function (pdf) on a space X and a one-to many mapping Γ from X to another space Y with the constraint that their focal elements $\Gamma(A)$ and $\Gamma(B)$ are disjoint: $\Gamma(A) \cap \Gamma(B) = \emptyset$ whenever $A \cap B = \emptyset$ with $A, B \subseteq X$. For our generalization of belief functions defined on the reals, we replace that constraint with the assumption that all focal elements are intervals[1] what leads to another model, none subsuming the other. The concept of linear belief functions can represent many types of knowledge, including linear equations, linear regression models, direct observations, full ignorance, normal distributions, etc... and its computation might be simpler and more efficient. Choosing between linear belief functions and belief functions on the reals depends on the application. Is the pdf on X 'meaningful'[2] and not just a mathematical abstraction for the linear belief functions? Can we justify the values of *bel* given to the intervals for the model presented in this paper? Do we need non-overlapping or interval-valued focal elements? These issues fall outside the scope of this study.

The paper is organized as follows. In section 2, we fix the notation we use to describe intervals. In section 3, we develop the theory of belief functions defined on $\mathcal{R}$. In section 4, we explain what become the rules describing the dynamic of beliefs in $\mathcal{R}$. In section 5, we introduce the concept of the bf-characteristic function and its potentiality for adding credal variables. In section 6, we introduce the concept of pignistic density functions. In section 7, we explain what is the q-Least Committed belief function induced by the knowledge of its pignistic density function. In section 8, we present the General Bayesian Theorem where the observation is defined on $\mathcal{R}$. In section 9, we conclude.

2 Intervals representation

2.1 Intervals of $\mathcal{R}$ and points in $\mathcal{R}^2$

We first define the extended real numbers which includes plus and minus infinity.

Definition 2.1 Extended real numbers. *The set $\mathcal{R} = \mathbb{R} \cup \{-\infty, \infty\}$ obtained by adjoining the two infinity elements to the set of real numbers $\mathbb{R}$ is*

[1] Some generalizations are manageable, see section 3.8.

[2] To be 'meaningful', a probability density function must concern a variable on which either bets could be established and settled (for the subjectivists) or frequencies of occurrence could be defined (for the frequentists).

called the set of extended real numbers.

In the next definitions, the symbols $\mathcal{I}$ and $\mathcal{T}$ hold for 'interval' and 'triangle'.

Definition 2.2 The $\mathcal{I}$ set. *Suppose $\alpha, \beta \in \mathcal{R}, \alpha < \beta$. We define:*

$$\mathcal{I}_{[\alpha,\beta]} = \{[x, y], (x, y], [x, y), (x, y) : x, y \in [\alpha, \beta]\}$$
$$\mathcal{I} = \{[x, y], (x, y], [x, y), (x, y) : x, y \in \mathcal{R}\}$$

as the set of closed, half open and open intervals in $[\alpha, \beta]$ or $\mathcal{R}$, respectively.

The set of intervals $\mathcal{I}$ on $\mathcal{R}$ contains the classical intervals of $I\!R$ among which $\emptyset$ and the intervals $[-\infty, y], [x, \infty]$ and $[-\infty, \infty]$. Note that $[x, y] = \emptyset$ whenever $x > y$.

Definition 2.3 The $\mathcal{T}$ set. *Closed intervals in $[\alpha, \beta]$ or $\mathcal{R}$ can be represented as points in an extended two dimensional space. We define:*

$$\mathcal{T}_{[\alpha,\beta]} = \{(x, y) : x, y \in [\alpha, \beta], x \leq y\},$$
$$\mathcal{T} = \{(x, y) : x, y \in \mathcal{R}, x \leq y\}.$$

The set $\mathcal{T}$ is also an extended set as it contains the infinities.

Figure 1 illustrates graphically this representation. The diagonal represents the domain $[0, 1]$. Any interval in $[0, 1]$ is represented by a point in the upper left triangle. So interval $[a, b] \subseteq [0, 1]$ is represented by the point K which coordinates are the upper side of the triangle, denoted 'from', and the left side of the triangle, denoted 'to'. The same representation can be adapted for the intervals of $\mathcal{I}_{[\alpha,\beta]}$ and $\mathcal{I}$.

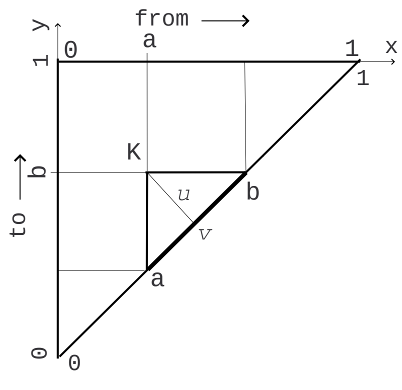

Figure 1: *Point $K = (a, b)$ inside the triangle $\mathcal{T}_{[0,1]}$, uniquely defines the interval* $[a, b] \subseteq [0, 1]$

The $\mathcal{U}$ set. Another very convenient representation consists in representing $[a, b] \in \mathcal{I}$ as a pair $(u, v) \in \mathcal{U}$ where $\mathcal{U} = \{(u, v) : u, v \in \mathcal{R}, u \geq 0\}$, where u is the distance from $(a, b) \in \mathcal{T}$ to the perpendicular projection of (a, b) on the diagonal $\mathcal{R}$ that contains the intervals, and v is the coordinate of this projection along this $\mathcal{R}$ diagonal (see figure 1).

Relations between $\mathcal{T}$ and $\mathcal{U}$ are given by:

$$u = (b-a)/2 \qquad a = v - u$$
$$v = (b+a)/2 \qquad b = v + u$$

By construction, the non empty elements of $\mathcal{I}$ are in one-to-one correspondence with those of $\mathcal{T}$ and $\mathcal{U}$ except v is undefined in the $\mathcal{U}$ representation for $[-\infty, \infty] \in \mathcal{I}$. For the empty set, $\emptyset \in \mathcal{I}$ but it is neither representable in $\mathcal{T}$ nor in $\mathcal{U}$. The most general representation is achieved with $\mathcal{I}$. The other two are sometimes more convenient, hence their introduction.

2.2 Dirac's and step functions

In our presentation, we repeatedly use the concepts of the step functions and Dirac's functions[3].

Definition 2.4 The step function. *The step function $H(x-x_0)$ centered at x_0 is a function defined by*

$$H(x-x_0) = \begin{cases} 0 & \text{if } x < x_0 \\ 1 & \text{if } x \geq x_0 \end{cases}$$

In particular, $H(a-x) = 1$ means $a \geq x$ and $1 - H(a-x) = 1$ means $a < x$.

Definition 2.5 The Dirac's function. *The Dirac's delta function (Dirac's function for short) $\delta(x-x_0)$ is a generalized function that is 0 everywhere except at its center x_0 where it is 'infinite' and with the property that $\int_{-\infty}^{\infty} f(x)\delta(x-x_0)dx = f(x_0)$.*

A Dirac's function is sometimes defined as the limit of a gaussian pdf of mean x_0 where the variance tends to 0, whereas the step function can be seen as the limit of the cumulative distribution functions related to the same pdfs.

Dirac's functions satisfy $\int_a^b \delta(x-b)dx = 1$ whereas $\int_a^{b-\varepsilon} \delta(x-b)dx = 0$, $\forall \varepsilon > 0$.

Dirac's and step functions are linked by:

Theorem 2.1

$$\frac{dH(x)}{dx} = \delta(x).$$

Probability functions can be represented as the sum of three components, an absolutely continuous one, a discrete one, and a singular one. For instance, the Gaussian pdf is an absolutely continuous component, a sum of weighted Dirac's functions is a discrete component, and the Cantor function is a singular component. As in probability theory, the third component is assumed to be absent in this paper. The discrete component can be represented by a sum of weighted Dirac's functions, which are handled as 'continuous' and 'derivable'.

Step functions can be used to define inclusion and compatibility functions. Their use can simplify some integrations.

[3]See Eric W. Weisstein. 'Heaviside Step Function' and 'Delta Function'. From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/HeavisideStepFunction.html and /DeltaFunction.html

Fct	Meaning	Constraints	Representation
$I_{[x,y]}^{[a,b]}$	$[x,y] \subseteq [a,b]$	$x \geq a$ & $y \leq b$	$H(x-a)H(b-y)$
$I_{(x,y)}^{[a,b]}$	$(x,y) \subseteq [a,b]$	$x \geq a$ & $y \leq b$	$H(x-a)H(b-y)$
$I_{[x,y]}^{(a,b]}$	$[x,y] \subseteq (a,b]$	$x > a$ & $y \leq b$	$(1-H(a-x))H(b-y)$
$I_{[x,y]}^{[a,b)}$	$[x,y] \subseteq [a,b)$	$x \geq a$ & $y < b$	$H(x-a)(1-H(y-b))$
$I_{[x,y]}^{(a,b)}$	$[x,y] \subseteq (a,b)$	$x > a$ & $y < b$	$(1-H(a-x))(1-H(y-b))$
$C_{[x,y]}^{[a,b]}$	$[x,y] \cap [a,b] \neq \emptyset$	$x \leq b$ & $y \geq a$	$H(b-x)H(y-a)$
$C_{[x,y]}^{(a,b]}$	$[x,y] \cap (a,b] \neq \emptyset$	$x \leq b$ & $y > a$	$H(b-x)(1-H(a-y))$
$C_{[x,y]}^{[a,b)}$	$[x,y] \cap [a,b) \neq \emptyset$	$x < b$ & $y \geq a$	$(1-H(x-b))H(y-a)$
$C_{[x,y]}^{(a,b)}$	$[x,y] \cap (a,b) \neq \emptyset$	$x < b$ & $y > a$	$(1-H(x-b))(1-H(a-y))$

Table 1: The representations of some inclusion and compatibility functions when all the intervals are non empty. I_A^B means $A \subseteq B$ and C_A^B means $A \cap B \neq \emptyset$.

Definition 2.6 The inclusion function. *For $A, B \in \mathcal{I}$, the inclusion function I_A^B is defined so that: $I_A^B = 1$ if $A \subseteq B$ and $I_A^B = 0$ otherwise.*

Definition 2.7 The compatibility function. *For $A, B \in \mathcal{I}$, the compatibility function C_A^B is defined so that: $C_A^B = 1$ if $A \cap B \neq \emptyset$ and $C_A^B = 0$ otherwise.*

In particular, $I_\emptyset^B = 1$ for all $B \in \mathcal{I}$, $I_A^\emptyset = 1$ iff $A = \emptyset$, $C_A^B = C_B^A$, and $C_A^\emptyset = 0$ for all $A \in \mathcal{I}$.

Both functions can be represented with step functions. A list of some of them is presented in table 1. For instance, $I_{[x,y]}^{(a,b]} = 1$ iff $[x,y] \subseteq (a,b]$, thus iff $x > a$ & $y \leq b$. One has $1 - H(a-x) = 1$ iff $x > a$ and $H(b-y) = 1$ iff $y \leq b$, hence $I_{[x,y]}^{(a,b]} = (1 - H(a-x))H(b-y)$.

3 Belief functions on $\mathcal{R}$

We fix the notation used for representing belief functions (section 3.1). Then for simplicity sake, we first define bbds on $\mathcal{R}$ in the case where there are only a finite number of focal elements (section 3.2). We then define belief function theory on the intervals on the extended reals (section 3.3). We formalize the nature of the frame of discernment (section 3.4) and list several special belief functions (section 3.5). We present the concept of belief discounting (section 3.6) and belief ordering (section 3.7). We finish this section by discussing generalization of the theory to n-dimensional frames (section 3.8).

3.1 Notation

In the classical TBM where the domain is finite, we use the next notation for the basic belief assignment (bba) m and its related functions b, bel, pl and q:

$$m^{\text{domain}}[\text{condition}](\text{subset}).$$

The three parameters denote respectively:

- domain : the set of elements on which the bba m is defined,
- condition : the condition which is accepted as true by the belief holder when he/she assesses the bba m,
- subset : any subset of the domain.

For instance, $bel^{\Omega}[Ev](A) = .6$ means that the belief holder allocates a belief .6 to the fact that the actual world belongs to the subset $A \subseteq \Omega$ given the belief holder accepts Ev as true.

So $m^{\Omega}[Ev]$ is the basic belief assignment (bba), a mapping from Ω to $[0, 1]$, whereas $m^{\Omega}[Ev](A)$ is the value taken by the bba at $A \subseteq \Omega$ and is called the basic belief mass (bbm) allocated to A. A subset of the domain where the bbm is positive is called a focal element.

In the continuous case, the equivalent of the bbas will become densities, called the basic belief densities (bbd), and their range will be $[0, \infty)$ (see section 3.3).

3.2 Finite number of focal elements

Let $\mathcal{A}$ be a finite collection of intervals in $[\alpha, \beta]$: $\mathcal{A} = \{A_i : A_i \in \mathcal{I}_{[\alpha,\beta]}, i = 1, ..., n\}$. Consider a bba $m^{\mathcal{A}} : \mathcal{A} \to [0, 1]$ which satisfies $\sum_{i=1,...,n} m^{\mathcal{A}}(A_i) = 1$. The A_i's with $m^{\mathcal{A}}(A_i) > 0$ are the focal elements of the bba $m^{\mathcal{A}}$.

On each point (a, b) in $\mathcal{T}_{[\alpha,\beta]}$ that corresponds to a focal element $[a, b] \in \mathcal{I}_{[\alpha,\beta]}$ of $m^{\mathcal{A}}$, we put a mass equal to $m^{\mathcal{A}}([a, b])$.

The end result of this mass allocation is a probability density function (pdf) $f^{\mathcal{T}_{[\alpha,\beta]}}$ on $\mathcal{T}_{[\alpha,\beta]}$ made of weighted Dirac's functions. We have for every $(x, y) \in \mathcal{T}_{[\alpha,\beta]}$:

$$f^{\mathcal{T}_{[\alpha,\beta]}}(x, y) = \sum_{i=1,...,n} m^{\mathcal{A}}([a_i, b_i])\delta(x - x_i)\delta(y - y_i),$$

where $A_i = [a_i, b_i]$.

Example 1. In table 3.2, we present an example with six focal elements in $[\alpha, \beta] = [0, 1]$. Figure 2 displays these focal elements. □

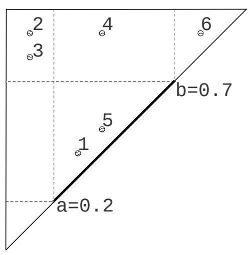

Figure 2: Graphical representation of the focal elements corresponding to table 3.2 data.

Given $m^{\mathcal{A}}$, we can define its related functions $bel^{\mathcal{A}}, pl^{\mathcal{A}}, q^{\mathcal{A}}$. Let $X = [a, b]$ be an interval in $[\alpha, \beta]$.

		$A_i = [a_i, b_i]$		$A = [.2, .7]$		
i	$m^{\mathcal{A}}$	a_i	b_i	$bel^{\mathcal{A}}$	$q^{\mathcal{A}}$	$pl^{\mathcal{A}}$
1	.07	.3	.4	$\times$		$\times$
2	.18	.1	.9		$\times$	$\times$
3	.25	.1	.8		$\times$	$\times$
4	.15	.4	.9			$\times$
5	.05	.4	.5	$\times$		$\times$
6	.30	.8	.9			
total	1.00			.12	.43	.70

Table 2: Bba defined on $[0, 1]$ with a finite number of focal sets. The $\times$ in the last three columns indicate the masses included in $bel^{\mathcal{A}}, q^{\mathcal{A}}$ and $pl^{\mathcal{A}}$.

Belief function. $bel^{\mathcal{A}}(X)$ is the sum of the masses given to the subsets of $X = [a, b]$, thus to the non empty intervals $A_i = [a_i, b_i]$ where $[a_i, b_i] \subseteq [a, b]$, thus $a_i \geq a, b_i \leq b$. In figure 2, one draws an horizontal and a vertical line from point (a, b) toward the diagonal line. This creates the triangle shown in figure 3.a. Every mass included in $bel^{\mathcal{A}}(X)$ must be in this triangle as it contains all the intervals $[x, y]$ where $x \geq a, y \leq b$ and only them. To get $bel^{\mathcal{A}}(X)$ one adds the masses of the focal elements located on the triangle. In our example $bel^{\mathcal{A}}([.2, .7]) = .12$.

Commonality function. $q^{\mathcal{A}}(X)$ for $X = [a, b]$, is defined as the sum of the masses given to the intervals $A_i = [a_i, b_i]$ where $[a, b] \subseteq [a_i, b_i]$, thus $a \geq a_i, b \leq b_i$. In figure 2, one draws an horizontal line from point (a, b) toward the left border of $\mathcal{T}_{[0,1]}$, and a vertical line from (a, b) up to the upper border of $\mathcal{T}_{[0,1]}$, defining thus a rectangle shown in figure 3.b. Every mass included in $q^{\mathcal{A}}(X)$ must be in this rectangle as it contains all the intervals $[x, y]$ where $x \leq a, y \geq b$ and only them. To get $q^{\mathcal{A}}(X)$ one adds the masses of the focal elements located on the rectangle just defined. In our example $q^{\mathcal{A}}([.2, .7]) = .43$.

Plausibility function. $pl^{\mathcal{A}}(X)$ for $X = [a, b]$, is defined as the sum of the masses given to the intervals $A_i = [a_i, b_i]$ where $[a, b] \cap [a_i, b_i]$, thus $a \leq b_i, b \geq a_i$. In figure 2, one uses the triangle built for $bel^{\mathcal{A}}(X)$, draws an horizontal line from its lower corner up to the left border of $\mathcal{T}_{[0,1]}$, and a vertical line from its upper corner up to the upper border of $\mathcal{T}_{[0,1]}$, delimiting so an area shown in figure 3.c. Every mass included in $pl^{\mathcal{A}}(X)$ must be in the area just defined as it contains all the intervals $[x, y]$ where $x \leq b, y \geq a$ and only them. To get $pl^{\mathcal{A}}(X)$ one adds the masses of the focal elements located on the area just defined. In our example $pl^{\mathcal{A}}([.2, .7]) = .70$. □

3.3 Basic belief densities

We can relax the fact that that the intervals belong to a bounded interval and the number of focal elements is finite, or even countable. The bounded interval domain is replaced by $\mathcal{R}$, $\mathcal{I}_{[\alpha,\beta]}$ becomes $\mathcal{I}$ and $\mathcal{T}_{[\alpha,\beta]}$ becomes $\mathcal{T}$. Everything

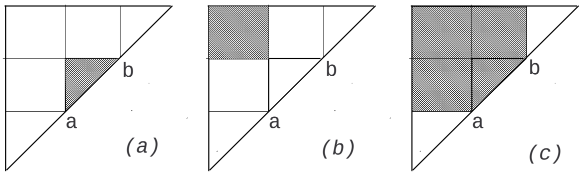

Figure 3: *Graphical representation of (a) belief; (b) commonality; (c) plausibility.*

described up to here will be essentially similar, masses become densities and sums become integrals[4].

For notational simplicity, we use the next symbols:

$$\iint_{x,y} \ldots dydx = \int_{x=-\infty}^{x=\infty} \int_{y=-\infty}^{y=\infty} \ldots dydx$$

where the indexes of the double integrals are the integration variables and their domain is the whole extended real line.

We generalize the classical bba into a 'basic belief density' (bbd) on $\mathcal{I}$. This function $m^{\mathcal{I}}$ plays the role of the bba except now it is a density, not a mass, hence its name.

Definition 3.1 The basic belief density. *A basic belief density $m^{\mathcal{I}}$ is a non negative function on $\mathcal{I}$ such that $m^{\mathcal{I}}(A) = 0$ if A is not a closed interval in $\mathcal{I}$ or $\emptyset$ and*

$$INT = \int_{x=-\infty}^{x=\infty} \int_{y=x}^{y=\infty} m^{\mathcal{I}}([x,y])dydx = \iint_{x,y} m^{\mathcal{I}}([x,y])H(y-x)dydx \leq 1.$$

We define:

$$m^{\mathcal{I}}(\emptyset) = 1 - INT.$$

Definition 3.2 The focal elements. *The elements A of $\mathcal{I}$ such that $m^{\mathcal{I}}(A) > 0$ are called the focal elements of $\mathcal{I}$.*

In this definition of the bbd, all focal elements are closed intervals or $\emptyset$. This choice is of course a matter of convenience. We could have used half open or open intervals. In the absence of Dirac's functions, the choice is irrelevant.

Given a normalized bbd $m^{\mathcal{I}}$ (i.e., $m^{\mathcal{I}}(\emptyset) = 0$), we can define another function $f^{\mathcal{I}}$ on $\mathcal{R}^2$ where $f^{\mathcal{I}}(a,b) = m^{\mathcal{I}}([a,b])$ for $a \leq b$ and $f^{\mathcal{I}}(a,b) = 0$ whenever $a > b$. $f^{\mathcal{I}}$ is a probability density function (pdf) on $\mathcal{R}^2$. When $m^{\mathcal{I}}$ is not normalized bbd (i.e., $m^{\mathcal{I}}(\emptyset) > 0$), the integral of $f^{\mathcal{I}}$ on its domain is $INT = 1 - m^{\mathcal{I}}(\emptyset)$. By abuse of language, we will still call it a pdf.

[4]For simplicity sake, we use Riemann integrals, but Lebesgue integrals could as well be used. Besides all integrals are applied on continuous pdfs with the understanding that Dirac's and step functions are 'continuous'.

Definition 3.3 Probability density function. *The function $f^{\mathcal{T}}$ defined on $\mathcal{R}^2$ such that for all $a, b \in \mathcal{R}$:*

$$\begin{aligned} f^{\mathcal{T}}(a,b) &= m^{\mathcal{I}}([a,b]), & \text{if } a \leq b \\ &= 0 & \text{if } a > b, \end{aligned}$$

or equivalently $f^{\mathcal{T}}(a,b) = m^{\mathcal{I}}([a,b])H(b-a)$, is called a probability density function (pdf).

The case where the domain is a finite interval $[\alpha, \beta]$ is covered by this general case by taking $f^{\mathcal{T}}(x,y) = 0$ whenever $(x,y) \notin \mathcal{T}_{[\alpha,\beta]}$.

The presence of Dirac's functions in the probability density function $f^{\mathcal{T}}$ is often a nuisance and in many cases they are absent in which case $f^{\mathcal{T}}$ is said to be absolutely continuous.

Definition 3.4 Absolutely continuous bbd. *An absolutely continuous bbd $m^{\mathcal{I}}$ is a bbd which related $f^{\mathcal{T}}$ probability density function is absolutely continuous, i.e., has no Dirac's functions.*

Just as in the previous section, we define the related $bel^{\mathcal{I}}, pl^{\mathcal{I}}, q^{\mathcal{I}}$ and $b^{\mathcal{I}}$ functions[5]. They become integrals of $f^{\mathcal{T}}$ on the various surfaces already described in section 3.2. We have the next definitions for the intervals (the other cases are covered by theorem 3.4):

Definition 3.5 Related functions. *For all $a \leq b, a, b \in \mathcal{R}$,*

belief function

$$bel^{\mathcal{I}}([a,b]) = \int_{x=a}^{x=b} \int_{y=x}^{y=b} m^{\mathcal{I}}(x,y)dydx, \qquad bel^{\mathcal{I}}(\emptyset) = 0 \tag{1}$$

plausibility function

$$pl^{\mathcal{I}}([a,b]) = \int_{x=-\infty}^{x=b} \int_{y=\max(a,x)}^{y=\infty} m^{\mathcal{I}}(x,y)dydx, \qquad pl^{\mathcal{I}}(\emptyset) = 0 \tag{2}$$

commonality function

$$q^{\mathcal{I}}([a,b]) = \int_{x=-\infty}^{x=a} \int_{y=b}^{y=\infty} m^{\mathcal{I}}(x,y)dydx, \qquad q^{\mathcal{I}}(\emptyset) = 1 \tag{3}$$

implicability function

$$b^{\mathcal{I}}([a,b]) = bel^{\mathcal{I}}([a,b]) + m^{\mathcal{I}}(\emptyset) \qquad b^{\mathcal{I}}(\emptyset) = m^{\mathcal{I}}(\emptyset). \tag{4}$$

Note that when $a = b$, Dirac's functions centered on a are included in each function as can be observed by taking the interval $[a,a]$ as the limit of the intervals $[a-\epsilon, a+\epsilon]$ where $\epsilon \to 0$.

In particular, $bel^{\mathcal{I}}([-\infty,\infty]) = pl^{\mathcal{I}}([-\infty,\infty]) = 1 - m^{\mathcal{I}}(\emptyset)$, $q^{\mathcal{I}}([-\infty,\infty]) = m^{\mathcal{I}}([-\infty,\infty])$ and $b^{\mathcal{I}}([-\infty,\infty]) = 1$.

These definitions can also be expressed using the inclusion and compatibility functions (see section 2.2). Remember that $f^{\mathcal{T}}(x,y) = 0$ whenever $x > y$.

[5]In general, the implicability function b is defined as $bel + m(\emptyset)$.

Theorem 3.1 *For all $a, b \in \mathcal{R}$, and*

$$bel^{\mathcal{I}}([a,b]) = \iint_{x,y} m^{\mathcal{I}}([x,y])H(y-x)I_{[x,y]}^{[a,b]}dydx = \iint_{x,y} I_{[x,y]}^{[a,b]} f^{\mathcal{T}}(x,y)dydx, \quad (5)$$

$$pl^{\mathcal{I}}([a,b]) = \iint_{x,y} m^{\mathcal{I}}([x,y])H(y-x)C_{[x,y]}^{[a,b]}dydx = \iint_{x,y} C_{[x,y]}^{[a,b]} f^{\mathcal{T}}(x,y)dydx, \quad (6)$$

$$q^{\mathcal{I}}([a,b]) = \iint_{x,y} m^{\mathcal{I}}([x,y])H(y-x)I_{[a,b]}^{[x,y]}dydx = \iint_{x,y} I_{[a,b]}^{[x,y]} f^{\mathcal{T}}(x,y)dydx, \quad (7)$$

When defining these functions on (half) open intervals, the last representation is simpler. E.g., compare relation (5) with what we would need for the first representation:

$$bel^{\mathcal{I}}((a,b)) = \int_{x=a+}^{x=b-} \int_{y=x}^{y=b-} f^{\mathcal{T}}(x,y)dydx.$$

Both are correct, but the use of the inclusion and compatibility functions simplifies the proofs of many theorems.

These definitions can also be expressed using the $\mathcal{U}$ representation of the intervals. Let $g^{\mathcal{U}}$ denote the function that expresses the densities on the $\mathcal{U}$ space, with $g^{\mathcal{U}}(u,v) = 2f^{\mathcal{T}}(v-u, v+u)$.

Theorem 3.2 *For all $a \leq b, a, b \in \mathcal{R}$,*

$$bel^{\mathcal{I}}([a,b]) = \int_{u=0}^{u=(b-a)/2} \int_{v=a+u}^{v=b-u} g^{\mathcal{U}}(u,v)dvdu, \qquad bel^{\mathcal{I}}(\emptyset) = 0 \quad (8)$$

$$pl^{\mathcal{I}}([a,b]) = \int_{u=0}^{u=\infty} \int_{v=a-u}^{v=b+u} g^{\mathcal{U}}(u,v)dvdu, \qquad pl^{\mathcal{I}}(\emptyset) = 0 \quad (9)$$

$$q^{\mathcal{I}}([a,b]) = \int_{u=(b-a)/2}^{u=\infty} \int_{v=b-u}^{v=a+u} g^{\mathcal{U}}(u,v)dvdu, \qquad q^{\mathcal{I}}(\emptyset) = 1. \quad (10)$$

We can derive $f^{\mathcal{T}}$ from relations (1) or (5) and (3) or (7) . We provide the proof for the first case in order to illustrate how the derivatives work.

Theorem 3.3 *If the derivatives exist,*

$$f^{\mathcal{T}}(a,b) = -\frac{\partial^2 bel^{\mathcal{I}}([a,b])}{\partial a \partial b}, \quad (11)$$

$$f^{\mathcal{T}}(a,b) = -\frac{\partial^2 q^{\mathcal{I}}([a,b])}{\partial a \partial b}. \quad (12)$$

Proof. Using relation (1) and table 1 data,

$$\frac{\partial^2 bel^{\mathcal{I}}([a,b])}{\partial a \partial b} = \frac{\partial}{\partial a}\Big(\int_{y=b}^{y=b} f^{\mathcal{T}}(b,y)dy + \int_{x=a}^{x=b} f^{\mathcal{T}}(x,b)dx \Big)$$
$$= 0 - f^{\mathcal{T}}(a,b).$$

Using relation(5), we get:

$$\begin{aligned}
bel^{\mathcal{I}}([a,b]) &= \iint\limits_{x,y} I_{[x,y]}^{[a,b]} f^{\mathcal{I}}(x,y)dydx \\
&= \iint\limits_{x,y} H(x-a)H(b-y)f^{\mathcal{I}}(x,y)dydx
\end{aligned}$$

Taking derivatives on a and b, we get:

$$\begin{aligned}
\frac{\partial^2 bel^{\mathcal{I}}([a,b])}{\partial a \partial b} &= \iint\limits_{x,y} \frac{\partial^2 H(x-a)H(b-y)}{\partial a \partial b} f^{\mathcal{I}}(x,y)dydx \\
&= \iint\limits_{x,y} -\delta(x-a)\delta(b-y)f^{\mathcal{I}}(x,y)dydx \\
&= -f^{\mathcal{I}}(a,b)
\end{aligned}$$

□

Because positive bbds are given only to closed intervals and $\emptyset$, these belief functions satisfy a limited form of additivity described in the next theorem. This theorem permits to extend the definition of $bel^{\mathcal{I}}$ and $q^{\mathcal{I}}$ on sets of intervals.

Theorem 3.4 *Suppose a bbd* $m^{\mathcal{I}}$ *and its related* $bel^{\mathcal{I}}$*. Let* $\{A_i = [a_i, b_i] \in \mathcal{I}, i = 1, 2, ...\}$ *be a collection of pairwise disjoint intervals in* $\mathcal{I}$*:*

$$A_{i_1} \cap A_{i_2} = \emptyset, i_1, i_2 \in \{1, 2, ...\}, i_1 \neq i_2.$$

Then:

$$bel^{\mathcal{I}}(\cup_{i=1,2,...} A_i) = \sum_{i=1,2,...} bel^{\mathcal{I}}(A_i), \tag{13}$$

$$q^{\mathcal{I}}(\cup_{i=1,2,...} A_i) = q^{\mathcal{I}}([\wedge_i a_i, \vee_i b_i])). \tag{14}$$

where $\vee$ *and* $\wedge$ *denote the max and min functions, respectively.*

Proof. By construction, $m^{\mathcal{I}}(A) = 0$ whenever A is not an interval. Let A_1 and A_2 be two disjoint intervals. $bel^{\mathcal{I}}(A_1 \cup A_2)$ contains all densities given to subsets of A_1 (thus included in $bel^{\mathcal{I}}(A_1)$), all densities given to subsets of A_2 (thus included in $bel^{\mathcal{I}}(A_2)$), and all densities that are given to subsets of $A_1 \cup A_2$ and which are neither subsets of A_1 nor of A_2. These last densities are thus given to subsets B with $B \cap A_1 \neq \emptyset$, $B \cap A_2 \neq \emptyset$, and $B \cap (A_1 \cup A_2)^c = \emptyset$. Such subsets that have elements in both A_1 and A_2 but not in between are not intervals, hence their densities are null. The only densities that remain in $bel^{\mathcal{I}}(A_1 \cup A_2)$ are all those included in $bel^{\mathcal{I}}(A_1)$ and $bel^{\mathcal{I}}(A_2)$. The proof is the same for any number of pairwise disjoint intervals, hence relation(13).

For the q relation (14), the densities that enter in the left term are those given to intervals that cover the union $\cup_{i=1,2,...} A_i$, hence given to the supersets of $[\wedge_i a_i, \vee_i b_i]$, thus relation (14). □

By comparing the various definitions, we get the next theorems.

Theorem 3.5 *For any* $f^{\mathcal{I}}$,

$$
\begin{aligned}
&q^{\mathcal{I}}([a,a]) = pl^{\mathcal{I}}([a,a]) \\
&pl^{\mathcal{I}}([a,b]) = bel^{\mathcal{I}}([-\infty,\infty]) - bel^{\mathcal{I}}(\overline{[a,b]}) \\
&bel^{\mathcal{I}}([a,b]) = pl^{\mathcal{I}}([-\infty,\infty]) - pl^{\mathcal{I}}(\overline{[a,b]}) \\
&pl^{\mathcal{I}}([a,b]) = pl^{\mathcal{I}}([a,a]) + pl^{\mathcal{I}}([b,b]) + bel^{\mathcal{I}}((a,b)) - q^{\mathcal{I}}([a,b]), \ \forall [a,b] \in \mathcal{I}.
\end{aligned}
$$

Remember that $(a,a) = \emptyset$ and from relation (5), $bel^{\mathcal{I}}(\emptyset) = 0$.

Example 2. Laplace-Gamma bbd Suppose a bbd with $g^{\mathcal{U}}(u,v) = f(v : \alpha, \beta)\ h(u : \nu)$ where $f(v : \alpha, \beta)$ is a Laplace pdf with parameters $\alpha, \beta,\ \alpha \in (-\infty, \infty)$ and $\beta > 0$, and $h(u : \nu)$ is a gamma pdf with parameter $\nu > 0$.

$$f(v : \alpha, \beta) = \frac{1}{2\beta} e^{-|v-\alpha|/\beta},$$

$$h(u : \nu) = \frac{1}{\Gamma(\nu)} u^{\nu-1} e^{-u}.$$

Notations are based on (Abramowitz & Stegun, 1965, chapter 26).

$$bel([x,y]) = \int_{u=0}^{u=(y-x)/2} P([x+u, y-u)] : \alpha, \beta)\ h(u : \nu)\ du$$

where

$$P([x,y)] : \alpha, \beta) = \int_x^y f(v : \alpha, \beta)\ dv = \int_x^y \frac{1}{2\beta} e^{-|v-\alpha|/\beta}\ dv$$

Thus

$$P([x,y)] : \alpha, \beta) = \begin{cases} \frac{1}{2}(e^{(\alpha-x)/\beta} - e^{(\alpha-y)/\beta}) & \text{if } x \geq \alpha \\ 1 - \frac{1}{2}(e^{(\alpha-y)/\beta} + e^{(-\alpha+x)/\beta}) & \text{if } x < \alpha < y \\ \frac{1}{2}(e^{(-\alpha+y)/\beta} - e^{(-\alpha+x)/\beta}) & \text{if } y \leq \alpha \end{cases}$$

The integration over the gamma pdf must be done term by term. For instance, suppose $\alpha = 0, \beta = 1, x = 1, y = 3, \nu = 2$. We compute:

$$
\begin{aligned}
bel([1,3]) &= \int_{u=0}^{u=1} \frac{1}{2}(e^{-1-u} - e^{u-3}))ue^{-u} du \\
&= \frac{1}{2} \int_{u=0}^{u=1} (ue^{-1-2u} - ue^{-3}) du \\
&= \frac{1}{2}(\frac{e^{-1}}{4} \int_{w=0}^{w=2} we^{-w} dw - \frac{e^{-3}}{2}) \\
&= .0149
\end{aligned}
$$

□

3.4 The frame of discernment

In a setting with finite cardinality, the domain of the bbas and their related functions is the power set generated by a finite set Ω. When moving to the real domain like $\mathcal{R}$, the domain is limited to a special subset of the power set: the Borel sigma-algebra [6].

Formally, the domain of the bbds and their related belief functions, plausibility functions, implicability functions and commonality functions defined in this paper is the sigma-algebra generated by the elements of $\mathcal{I}$. Thus they map the elements of this Borel sigma-algebra to $[0, \infty)$.

The notations m^Ω for a finite case and $m^\mathcal{I}$ for the real space are coherent. The index of m is the set from which we build the Borel sigma-algebra. In the finite case, the Borel sigma-algebra generated by Ω is equal to the power set generated by Ω, hence the distinction is never mentioned. In the $\mathcal{R}$ case, the bbds and their related functions are limited to Borel sigma-algebra, not the whole power set. This is just what is done in probability theory.

3.5 Special bbds

3.5.1 Vacuous bbd

In order to use a unique symbol for the domain of a bbd, we use Ω to denote it. In the bounded domain case $\mathcal{I}_{[\alpha,\beta]}$, $\Omega = [\alpha, \beta]$. In the unbounded case $\mathcal{I}$, we use $\Omega = [-\infty, \infty]$.

Definition 3.6 *A vacuous bbd $m^\mathcal{I}$ is a bbd such that $m^\mathcal{I}(\Omega) = 1$.*

This bbd represents the state of total ignorance. No strict subset of Ω is supported.

3.5.2 Categorical bbd

Suppose all the belief holder knows is that the truth is in the interval $[a, b] \in \mathcal{I}$ with $[a, b] \neq \Omega$ where Ω is the domain of the bbd (see section 3.5.1). The bbd that represents such a belief state is represented by a categorical bbd, i.e., a Dirac's function centered at (a, b). The smallest the interval, the more precise the belief.

Definition 3.7 *A categorical bbd $m^\mathcal{I}$ is a bbd such that $m^\mathcal{I}([a, b]) = \delta(x-a, y-b)$ for $[a, b] \in \mathcal{I}$ and $[a, b] \neq \Omega$.*

3.5.3 Consonant bbd

Consider a bbd which focal elements are nested. With the $g^\mathcal{U}$ notation, it means there exists an index (usually continuous) such that the focal elements can be labeled by it as in $A(u)$. Then $A(u) \subseteq A(u')$ when $u' > u$.

Definition 3.8 *A consonant bbd is a bbd which focal elements are nested.*

[6] The Borel sigma-algebra on the set of real numbers is the sigma-algebra generated by the collection of closed intervals on the real numbers. As every sigma-algebra, it is closed under complementation, countable union and countable intersection. One can prove that it contains all open intervals, closed intervals, countably infinite unions or intersections of either.

With the $g^{\mathcal{U}}$ notation, we can express the next theorem in a simple way.

Theorem 3.6 *If bbd m is consonant, then given $u \in [0, \infty)$, there exists at most one $v \in [-\infty, \infty]$ with $g^{\mathcal{U}}(u, v) > 0$.*

Proof. Fix u and suppose two values v and v'. For the intervals to be nested, one must have $v - u \leq v' - u$ and $v + u \geq v' + u$ (or $v - u \geq v' - u$ and $v + u \leq v' + u$). In each case, it means $v = v'$. □

This theorem explains the interest of the (u, v) representation and why u can be taken as a convenient label for the nested elements.

In the triangle of figure 1, the nested nature implies that the density function is concentrated on a curve that leaves from the diagonal representing $\mathcal{I}$ and moves always in the upper left direction.

Theorem 3.7 *If bbd $m^{\mathcal{I}}$ is consonant, then $g^{\mathcal{U}}(u, v) = h(u)\delta(v - \psi(u))$ where $\psi(u)$ is the unique value for v given u when $h(u) > 0$ and such that $g^{\mathcal{U}}(u, v) > 0$.*

Proof. By theorem 3.6, u uniquely defines v, hence positive $g^{\mathcal{U}}$ is only a function of u. □

Theorem 3.8 *If bbd $m^{\mathcal{I}}$ is consonant and $u' > u$, then $\psi(u')$ is in the upper left quadrant centered on $(u, \psi(u))$.*

Proof. Fix u and $v = \psi(u)$. Let $u' > u$. The related intervals are $[v - u, v + u]$ and $[v' - u', v' + u']$. To be nested, the first must be a subset of the second, hence $v - u \geq v' - u'$ and $v + u \leq v' + u'$. Any point of $\mathcal{T}$ which is not in the upper left quadrant violates one of these two inequalities. □

3.5.4 The Bayesian belief functions

Probability density functions are special cases of belief functions where densities are given only to singletons. In our present context, it means that the density is concentrated on the diagonal itself (where intervals are degenerated into points). Such belief functions are called Bayesian belief functions.

Definition 3.9 *Let $f(v)$ be a pdf on $\mathcal{R}$. The Bayesian belief function on $\mathcal{I}$ based on f is a belief function with $g^{\mathcal{U}}(u, v) = \delta(u)f(v)$*

3.5.5 U-non-interaction

For practical applications, the bbd may often be represented by decomposable functions, like in the next case.

Definition 3.10 *A bbd is called a U-non-interactive bbd iff its related $g^{\mathcal{U}}$ function satisfies $g^{\mathcal{U}}(u, v) = h(u)f(v)$.*

The Bayesian belief function is an example of U–non-interaction.

Another practical example is obtained with $f(v) = N(v : \mu, \sigma)$ is Gaussian. If furthermore $h(u) \propto N(u : \nu, \eta)$ when $u \geq 0$, the resulting density corresponds to a censored bivariate gaussian distribution of two independent variables, where the part for $u < 0$ is made null. Another useful example is obtained when $h(u)$ is a gamma distribution (which includes the exponential pdf case).

3.6 Discounting

Suppose the belief holder has a vacuous a priori belief about a given variable and collects a bbd $m^{\mathcal{I}}$ relative to this variable from a source S.

Let Ω denote the variable domain.

If the belief holder accepts that S is fully reliable, he/she would consider $m^{\mathcal{I}}$ as representing his/her belief. If the belief holder accepts that S is absolutely not reliable, he/she would neglect $m^{\mathcal{I}}$, or equivalently consider that the bbd collected from S must be transformed into a vacuous bbd.

For non extreme cases, let $\alpha \in [0, 1]$ be the belief allocated by the belief holder to the fact that S is reliable. This case covers the two previous ones: $\alpha = 1$ means the source is accepted as fully reliable, and $\alpha = 0$ means the source is accepted as not reliable at all.

The impact of the partial reliability results in a discounting of the bbd $m^{\mathcal{I}}$ into a new bbd $m^{\mathcal{I},\alpha}$ with:

$$m^{\mathcal{I},\alpha}([a,b]) = \alpha m^{\mathcal{I}}([a,b]) \qquad \forall [a,b] \neq \Omega,$$
$$m^{\mathcal{I},\alpha}(\Omega) = \alpha m^{\mathcal{I}}(\Omega) + 1 - \alpha.$$

3.7 Ordering belief functions

Specialization. We define the notions of specialization (Dubois & Prade, 1986; Yager, 1986) within the classical finite frame used in belief function theory.

Suppose a bba m_1^{Ω} and another bba m_2^{Ω} obtained from the reallocation of every mass of m_1^{Ω} among the subset of its focal elements. So for every $A \in \Omega$, $m_1^{\Omega}(A)$ is reallocated among the subsets of A. In that case, we say that m_2^{Ω} is a specialization of m_1^{Ω}. Formal definition is presented in section 4.1.

Orderings. Dubois and Prade (1987) have proposed three solutions to order belief functions according to the 'strength' of the beliefs they represent. The intuitive idea is that the smaller the focal elements, the stronger the beliefs.

Let m_1 and m_2 be two bbas on Ω. Their proposals are:

- *pl-ordering.* If $pl_1(A) \leq pl_2(A)$ for all $A \subseteq \Omega$, we write $m_1 \sqsubseteq_{pl} m_2$
- *q-ordering.* If $q_1(A) \leq q_2(A)$ for all $A \subseteq \Omega$, we write $m_1 \sqsubseteq_q m_2$
- *s-ordering.* If m_1 is a specialization of m_2, we write $m_1 \sqsubseteq_s m_2$

When bbas are normalized, $m_1 \sqsubseteq_{pl} m_2$ implies $bel_1(A) \geq bel_2(A)$ for all $A \subseteq \Omega$.

They prove that :

- $m_1 \sqsubseteq_s m_2$ implies $m_1 \sqsubseteq_{pl} m_2$ and $m_1 \sqsubseteq_q m_2$, but the reverse is not true.
- $m_1 \sqsubseteq_{pl} m_2$ and $m_1 \sqsubseteq_q m_2$ do not imply each other.

In (Smets, 1983; Smets & Magrez, 1985), we define the 'information content' of a bba and use it for ordering belief functions. The q-ordering implies this ordering.

The s-ordering is thus stronger then the others as it implies them. Whenever $m_1 \sqsubseteq_X m_2$ for $X \in \{s, pl, q\}$, we say that m_2 is X-less committed (X-LC) than m_1. The same qualification is extended to the functions related to the bbas.

The concept of 'least commitment' permits the construction of a partial order $\sqsubseteq$ on the set of belief functions (Yager, 1986; Dubois & Prade, 1987).

The Principle of Minimal Commitment consists in selecting the least committed belief function in a set of equally justified belief functions. The principle formalizes the idea that one should never give more support than justified to any subset of Ω. It satisfies a form of skepticism, of noncommitment, of conservatism in the allocation of the beliefs. In its spirit, it is not far from what the probabilists try to achieve with the maximum entropy principle (Dubois & Prade, 1987; Hsia, 1991).

Which order should be used? The best candidate seems to be the s-ordering, as it implies the others. But when there is no s-least committed solution, the q-ordering seems to be appropriate, in particular because of the meaning of q.

The meaning of $q(A)$. When $\Omega = \{x, y\}$ the difference $pl(x) - bel(x)$ has often been proposed as a measure of the uncertainty in bel. In fact $pl(x) - bel(x) = m(\{x, y\})$ and $m(\{x, y\})$, as well as $m(\Omega)$ in general, is the part of belief free to flow anywhere, totally uncommitted. So to consider $m(\Omega)$ as the measure of uncertainty seems reasonable. Suppose now we accept that A is true where $A \subseteq \Omega$. Then $m[A](A)$ obtained by conditioning m with Dempster's rule of conditioning becomes the 'conditional measure of uncertainty' in context A. It just happens that $m[A](A) = q(A)$, so the commonality function is the set of conditional measures of uncertainty, and ordering beliefs according to q becomes very natural.

3.8 Generalization to $\mathcal{R}^n$

The real issue underlying the possibility to extend belief functions on $\mathcal{R}^n$ is the existence of a finite dimensional real space, the elements of which are in one-to-one relation with the focal elements. This can be done when all focal elements are ellipses, or rectangles, or hexagons...

If all focal elements can be so represented as a point in $\mathcal{R}^p$ for some $p > 0$, the theory extends directly. Of course integrals become high dimensional and difficult to manage.

4 Conjunctive belief revision

Beliefs held by the belief holder concerns the actual value of a given variable. Suppose two beliefs induced by two pieces of evidence that bear directly on the actual value of the variable, and produced by two sources. Suppose the belief holder accepts both sources as fully reliable. Then the two beliefs are to be conjunctively combined into a new belief that quantifies the impact of the two pieces of evidence on the actual value of the variable.

Identically, when the belief holder has some prior beliefs on the actual value of the variable and collects a piece of evidence from a new source accepted as fully reliable, the prior beliefs are revised in a conjunctive way.

In the TBM, both cases are usually treated in the same way (nevertheless see (Smets, 2004) for dynamic belief revision), and the belief functions are revised by some conjunctive revision. A very general form of conjunctive belief revision is represented by a specialization. Special forms of specializations are described by:

1. the conditioning process when one of the belief functions is categorical,
2. the conjunctive combination when the two pieces of evidence that induce the bbas are distinct.

We study successively these three forms of belief revisions.

4.1 Specialization

A specialization is a transformation that maps bbds into bbds and satisfies the next requirement. For every focal element $[x, y]$ of the first bbd $m_1^{\mathcal{I}}$, the density $m_1^{\mathcal{I}}([x, y])$ is distributed among the densities $m_2^{\mathcal{I}}([a, b])$ given by the second bbd to focal elements $[a, b]$ that are subsets of $[x, y]$. This process could be summarized by the expression 'masses flow down', what reflects that the new bbd is more informative than the initial one.

In $\mathcal{T}$, specializations are represented as follows.

Definition 4.1 Specialization operators. *A specialization operator $s^{\mathcal{T}}$ is a mapping $\mathcal{T} \times \mathcal{T} \to [0, \infty)$ that satisfies for all $[x, y] \in \mathcal{I}$:*

$$s^{\mathcal{T}}(a, b|x, y) = 0, \quad \textit{whenever } [a, b] \nsubseteq [x, y] \textit{ or } [a, b] = \emptyset, \tag{15}$$

$$\int_{a=x}^{a=y} \int_{b=a}^{b=y} s^{\mathcal{T}}(a, b|x, y) dbda = \iint\limits_{a,b} s^{\mathcal{T}}(a, b|x, y) I_{[a,b]}^{[x,y]} dbda \leq 1. \tag{16}$$

In fact, $s^{\mathcal{T}}(a, b|x, y)$ is an unnormalized probability density function on $\mathcal{T}_{[x,y]}$ (unnormalized as some mass can be given to the empty set).

Definition 4.2 Bbd specialization. *The specialization of the a bbd $m_1^{\mathcal{I}}$ is a bbd $m_2^{\mathcal{I}}$ which satisfies:*

$$f_2^{\mathcal{T}}(a, b) = \iint\limits_{x,y} s^{\mathcal{T}}(a, b|x, y) f_1^{\mathcal{T}}(x, y) dydx$$

where $s^{\mathcal{T}}$ is a specialization operator.

The term $s^{\mathcal{T}}(a, b|x, y)$ represents the 'part' of the bbd $m_1^{\mathcal{I}}$ given to $[x, y]$ that is transferred to the interval $[a, b]$ of the new bbd $m_2^{\mathcal{I}}$.

4.2 Belief conditioning

4.2.1 Mass transfer

Suppose a bbd $m^{\mathcal{I}}$ that represents the beliefs held by the belief holder about the actual value of a given variable. Suppose the belief holder learns then that the actual value is in $[c, d]$. The bbd given to $[a, b]$ is transferred to $[a, b] \cap [c, d] = [a \vee c, b \wedge d]$ (which may be empty).

Definition 4.3 Dempster's rule of conditioning. *Given* $m^{\mathcal{I}}$ *and* $[c,d] \in \mathcal{I}$*, for every focal set* $[a,b]$*, the density* $m^{\mathcal{I}}([a,b])$ *is transferred into* $[a,b] \cap [c,d]$*. This process is called the Dempsterian conditioning and the resulting bbd, denoted* $m^{\mathcal{I}}[[c,d]]([a,b])$ *is called the conditional bbd given* $[c,d]$*.*

In order to derive the induced relations, we consider first the result of the belief transfer on the commonality function.

Theorem 4.1 *Given the bbd* $m^{\mathcal{I}}$*, the result of the conditioning of its related commonality function* $q^{\mathcal{I}}$ *on* $[c,d]$ *is given by the next commonality function:*

$$q^{\mathcal{I}}[[c,d]]([a,b]) = \begin{cases} q^{\mathcal{I}}([a,b]) & \text{if } [a,b] \subseteq [c,d] \\ 0 & \text{if } [a,b] \not\subseteq [c,d]. \end{cases}$$

or equivalently:

$$q^{\mathcal{I}}[[c,d]]([a,b]) = q^{\mathcal{I}}([a,b]) I_{[a,b]}^{[c,d]}. \tag{17}$$

Proof. We study the value $q^{\mathcal{I}}[[c,d]]([a,b])$ given by the commonality function to $[a,b]$ after the conditioning of $m^{\mathcal{I}}$ on $[c,d]$.

1. Suppose $c \leq d$ and $[a,b] \subseteq [c,d]$. Take a focal element $[x,y]$ of $m^{\mathcal{I}}$.
   (a) Suppose $[a,b] \subseteq [x,y]$. The bbd given to $[x,y]$ belonged to $q^{\mathcal{I}}([a,b])$ and is transferred to $[x,y] \cap [c,d]$ which is still a superset of $[a,b]$, so it also belongs to $q^{\mathcal{I}}[[c,d]]([a,b])$.
   (b) Suppose $[a,b] \not\subseteq [x,y]$, then so is $[x,y] \cap [c,d]$. The bbd given to $[x,y]$ did not belong to $q^{\mathcal{I}}([a,b])$ and after its transfer, it still does not belong to $q^{\mathcal{I}}[[c,d]]([a,b])$.
2. Suppose $c \leq d$ and $[a,b] \not\subseteq [c,d]$. Take any focal element $[x,y]$ of $m^{\mathcal{I}}$. After its intersection with $[c,d]$, it cannot be a superset of $[a,b]$, so its bbd will not be included in $q^{\mathcal{I}}[[c,d]]([a,b])$. Hence $q^{\mathcal{I}}[[c,d]]([a,b]) = 0$.
3. Suppose $[c,d] = \emptyset$, conditioning is still possible but leads to a bbd with the whole mass 1 allocated to $\emptyset$, hence for all $[a,b] \neq \emptyset$, $q^{\mathcal{I}}[\emptyset]([a,b]) = 0$ and $q^{\mathcal{I}}[\emptyset](\emptyset) = 1$.

□

To derive the analogous relation for the bbd itself, we use relation (12) and the inclusion function (definition 2.6).

Theorem 4.2 *Given the bbd* $m^{\mathcal{I}}$*, the result of its conditioning on* $[c,d]$ *is given by the next bbd:*

$$m^{\mathcal{I}}[[c,d]]([a,b]) = \begin{cases} m^{\mathcal{I}}([a,b]) & \text{if } c < a \leq b < d \\ \int_{x=-\infty}^{x=c} m^{\mathcal{I}}([x,b])dx, & \text{if } c = a \leq b < d \\ \int_{y=d}^{y=\infty} m^{\mathcal{I}}([a,y])dy, & \text{if } c < a \leq b = d \\ q^{\mathcal{I}}([c,d]) & \text{if } c = a \leq b = d \\ 1 - pl^{\mathcal{I}}([c,d]) & \text{if } [a,b] = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

or equivalently:

$$f^{\mathcal{I}}[[c,d]]([a,b]) = \iint\limits_{x,y} f^{\mathcal{I}}(x,y)\delta(a - x \vee c)\delta(b - d \wedge y)dydx \qquad (18)$$

Proof. From relation (12), we have for $[a,b] \neq \emptyset$:

$$m^{\mathcal{I}}[[c,d]]([a,b]) = -\frac{\partial^2 q^{\mathcal{I}}[[c,d]]([a,b])}{\partial a \partial b}.$$

Using (17), the right hand term becomes for $[a,b] \neq \emptyset$:

$$\begin{aligned}
&-\frac{\partial^2 q^{\mathcal{I}}([a,b]) I_{[a,b]}^{[c,d]}}{\partial a \partial b} = \\
&-\frac{\partial}{\partial b}\Big(\frac{\partial q^{\mathcal{I}}([a,b])}{\partial a} H(a-c)H(d-b) + q^{\mathcal{I}}([a,b])\delta(a-c)H(d-b)\Big) = \\
&-\frac{\partial^2 q^{\mathcal{I}}([a,b])}{\partial a \partial b} H(a-c)H(d-b) + \frac{\partial q^{\mathcal{I}}([a,b])}{\partial a} H(a-c)\delta(d-b)\ldots \\
&-\frac{\partial q^{\mathcal{I}}([a,b])}{\partial b}\delta(a-c)H(d-b) + q^{\mathcal{I}}([a,b])\delta(a-c)\delta(d-b)
\end{aligned}$$

From relation (3), we get:

$$\begin{aligned}
\frac{\partial q^{\mathcal{I}}([a,b])}{\partial a} &= \int_{y=b}^{y=\infty} m^{\mathcal{I}}([a,y])dy \\
\frac{\partial q^{\mathcal{I}}([a,b])}{\partial b} &= -\int_{x=-\infty}^{x=a} m^{\mathcal{I}}([x,b])dx
\end{aligned}$$

Replacing the terms and using relation (12), we get the four first terms of the theorem. These four terms concerns all the densities given to focal elements that are compatible with $[c,d]$, thus those in $pl^{\mathcal{I}}([c,d])$. The other densities are transferred to $m^{\mathcal{I}}[[c,d]](\emptyset)$, hence the fifth term. The other terms of $m^{\mathcal{I}}[[c,d]]$ are null. □

We give the analogous equations for *bel* and *pl*. The proofs are similar to those for *m*.

Theorem 4.3 *Given the bbd $m^{\mathcal{I}}$, the result of conditioning its related $bel^{\mathcal{I}}$ on $[c,d]$ is given by the next conditional belief function.*

1. *For $[a,b] \subseteq [c,d]$, $[a,b] \neq \emptyset$, $bel^{\mathcal{I}}[[c,d]]([a,b]) = \ldots$*

$$\begin{cases}
bel^{\mathcal{I}}([a,b]) & \text{if } a > c, b < d, a \leq b \\
bel^{\mathcal{I}}([a,\infty]) - bel^{\mathcal{I}}([d,\infty]) & \text{if } a > c, b = d, a \leq b \\
bel^{\mathcal{I}}([-\infty,b]) - bel^{\mathcal{I}}([-\infty,c]) & \text{if } a = c, b < d, a \leq b \\
bel^{\mathcal{I}}([-\infty,\infty]) - bel^{\mathcal{I}}([d,\infty]) - bel^{\mathcal{I}}([-\infty,c]) & \text{if } a = c, b = d \\
0 & \text{if } [a,b] = \emptyset
\end{cases}$$

2. *For $[a,b] = \emptyset$, $bel^{\mathcal{I}}[[c,d]](\emptyset) = 0$.*

3. For the other $[a,b]$*, we have* $bel^{\mathcal{I}}[[c,d]]([a,b]) = bel^{\mathcal{I}}[[c,d]]([a,b]\cap[c,d])$*.*

Theorem 4.4 *One has:*

$$bel^{\mathcal{I}}[[c,d]]([a,b]) = bel^{\mathcal{I}}([a,b]\cup\overline{[c,d]}) - bel^{\mathcal{I}}(\overline{[c,d]})$$

Theorem 4.5 *Given the bbd* $m^{\mathcal{I}}$*, the result of conditioning its related* $pl^{\mathcal{I}}$ *on* $[c,d]$ *is given by the next conditional plausibility function:*

$$pl^{\mathcal{I}}[[c,d]]([a,b]) = pl^{\mathcal{I}}([a,b]\cap[c,d])$$

These results can also be obtained when conjunctively combining $m^{\mathcal{I}}$ with a categorical bbd centered on $[c,d]$ (see definition 3.7 and section 4.3).

4.2.2 Conditioning and specialization

The Dempster's rule of conditioning can also be described as a specialization.

Theorem 4.6 *Given* $[c,d]\in\mathcal{I}$*, the function* $\delta(a-c\vee x)\delta(b-d\wedge y)$ *in relation (18) is a specialization operator* $s^{\mathcal{I}}(a,b|x,y)$ *(null everywhere except when* $[a,b]=[c,d]\cap[x,y]$*). Thus* $m^{\mathcal{I}}[[c,d]]$ *is a specialization of* $m^{\mathcal{I}}$*.*

Proof. $\delta(a-c\vee x)\delta(b-d\wedge y)$ is non negative, 0 when $[a,b]\not\subseteq[x,y]$, and its integral on a,b is 1 for all $[x,y]$, hence it is a specialization operator. From relation (18), $m^{\mathcal{I}}[[c,d]]$ is thus a specialization of $m^{\mathcal{I}}$. □

This particular specialization operator is called a conditioning specialization.

The next result is very important within the TBM, as it provides a real justification for using Dempster's rule of conditioning (Klawonn & Smets, 1992). Conditioning turns out be a specialization that satisfies what we feel are natural requirements.

Theorem 4.7 *Let* $\mathcal{S}p_{[c,d]}(m^{\mathcal{I}})$ *be the set of specialization* $m^{\mathcal{I}*}$ *of* $m^{\mathcal{I}}$ *such that* $pl^{\mathcal{I}*}(\overline{[c,d]})=0$*. Its s-least committed element is the bbd* $m^{\mathcal{I}}[[c,d]]$ *computed from Dempster's rule of conditioning.*

Proof. Suppose the bbd $m^{\mathcal{I}}$. Let $s^{\mathcal{I}}$ be a specialization operator and m^* be the result of its application to $m^{\mathcal{I}}$. Then $f^*(a,b) = \iint_{x,y} s^{\mathcal{I}}(a,b|x,y)f^{\mathcal{I}}(x,y)dydx$. To get $pl^*(\overline{[c,d]})=0$, one must have $s^{\mathcal{I}}(a,b|x,y)=0$ for all $[a,b]$ such that $C_{[a,b]}^{\overline{[c,d]}}=1$. As $s^{\mathcal{I}}$ is a specialization operator, one has also for all $[x,y]\in\mathcal{I}$, $s^{\mathcal{I}}(a,b|x,y)>0$ only if $[a,b]\subseteq[x,y]$. Therefore $s^{\mathcal{I}}(a,b|x,y)>0$ only if $[a,b]\subseteq[x,y]\cap[c,d]$. The s-least committed specialization operator that satisfies this constraint is the one that puts for each $[x,y]$ a mass 1 on its largest possible interval, which is $[x,y]\cap[c,d]$. This specialization operator is the one of theorem 4.6. □

That $m^{\mathcal{I}}[[c,d]]$ is a specialization of $m^{\mathcal{I}}$ fits with the idea that specializations represent the impact of the conjunctive combination of beliefs. That $pl^{\mathcal{I}}[[c,d]](\overline{[c,d]})=0$ translates that the actual value of the variable considered by $m^{\mathcal{I}}$ is accepted to be in $[c,d]$. Selecting the s-least committed bbd translates the idea that we should never give more beliefs than justified (the TBM concerns beliefs, not faith).

4.3 Conjunctive combination rule

4.3.1 Mass transfer

Suppose two belief functions $m_1^{\mathcal{I}}$ and $m_2^{\mathcal{I}}$ induced by two distinct pieces of evidence.

For the conjunctive rule of combination, the product

$$m_1^{\mathcal{I}}([a_1, b_1])m_2^{\mathcal{I}}([a_2, b_2])$$

is allocated to the interval $[a_1, b_1] \cap [a_2, b_2] = [a_1 \vee a_2, b_1 \wedge b_2]$ which may be empty.

Definition 4.4 *Suppose two bbds $m_1^{\mathcal{I}}$ and $m_2^{\mathcal{I}}$. Let*

$$\begin{aligned} m_{1 \bigcirc\!\!\!\!\cap\, 2}^{\mathcal{I}}([a,b]) = & \int_{x=-\infty}^{a} \int_{y=b}^{\infty} m_1^{\mathcal{I}}([x,b]) m_2^{\mathcal{I}}([a,y]) dy dx \\ & + \int_{x=-\infty}^{a} \int_{y=b}^{\infty} m_1^{\mathcal{I}}([a,y]) m_2^{\mathcal{I}}([x,b]) dy dx \\ & + m_1^{\mathcal{I}}([a,b]) \int_{x=-\infty}^{a} \int_{y=b}^{\infty} m_2^{\mathcal{I}}([x,y]) dy dx \\ & + m_2^{\mathcal{I}}([a,b]) \int_{x=-\infty}^{a} \int_{y=b}^{\infty} m_1^{\mathcal{I}}([x,y]) dy dx. \end{aligned}$$

or equivalently:

$$f_{1 \bigcirc\!\!\!\!\cap\, 2}^{\mathcal{I}}(a,b) = \iint_{w,z} \iint_{x,y} f_1^{\mathcal{I}}(w,z) f_2^{\mathcal{I}}(x,y) \delta(a - w \vee x) \delta(b - z \wedge y) dy dx dz dw \quad (19)$$

We say that the bbd $m_{1 \bigcirc\!\!\!\!\cap\, 2}^{\mathcal{I}}$ results from the application of the conjunctive combination rule.

In practice, when Dirac's functions are present, the easiest way to handle such 'messy' cases would be to separate the absolutely continuous part from the discrete part and to perform the computation on each part separately. Computation for the discrete part is identical to the one described for belief functions defined on the finite frames.

The relations among the commonality functions apply as in the finite cardinality case.

Theorem 4.8 *Suppose two bbds $m_1^{\mathcal{I}}$ and $m_2^{\mathcal{I}}$, and their conjunctive combination $m_{1 \bigcirc\!\!\!\!\cap\, 2}^{\mathcal{I}}$. Assuming the derivatives used in theorem 3.3 exist, their related commonality functions satisfy:*

$$q_{1 \bigcirc\!\!\!\!\cap\, 2}^{\mathcal{I}}([a,b]) = q_1^{\mathcal{I}}([a,b]) q_2^{\mathcal{I}}([a,b]), \ \forall [a,b] \in \mathcal{I}. \quad (20)$$

Proof. For simplicity sake, we omit the $\mathcal{I}$ index. Using relation (12), we get:

$$\begin{aligned}
m_{1 \circledcap 2}([a,b]) &= -\frac{\partial^2 q_{1 \circledcap 2}([a,b])}{\partial a \partial b} \\
&= -\frac{\partial^2 (q_1([a,b]) q_2([a,b]))}{\partial a \partial b} \\
&= -\frac{\partial^2 q_1([a,b])}{\partial a \partial b} q_2([a,b]) - q_1([a,b]) \frac{\partial^2 q_2([a,b])}{\partial a \partial b} \\
&- \frac{\partial q_1([a,b])}{\partial a} \frac{\partial q_2([a,b])}{\partial b} - \frac{\partial q_1([a,b])}{\partial b} \frac{\partial q_2([a,b])}{\partial a} \\
&= m_1([a,b]) q_2([a,b]) + q_1([a,b]) m_2([a,b]) \\
&- \frac{\partial q_1([a,b])}{\partial a} \frac{\partial q_2([a,b])}{\partial b} - \frac{\partial q_1([a,b])}{\partial b} \frac{\partial q_2([a,b])}{\partial a}
\end{aligned}$$

The first two terms are the last two of definition 4.4. For the other two, we use:

$$\begin{aligned}
\frac{\partial q([a,b])}{\partial a} &= \frac{\partial \int_{x=-\infty}^{a} \int_{b}^{\infty} m([x,y]) dy dx}{\partial a} &= \int_{y=b}^{\infty} m([a,y]) dy \\
\frac{\partial q([a,b])}{\partial b} &= \frac{\partial \int_{x=-\infty}^{a} \int_{b}^{\infty} m([x,y]) dy dx}{\partial b} &= -\int_{x=-\infty}^{a} m([x,b]) dx
\end{aligned}$$

Hence:

$$-\frac{\partial q_1([a,b])}{\partial a} \frac{\partial q_2([a,b])}{\partial b} - \frac{\partial q_1([a,b])}{\partial b} \frac{\partial q_2([a,b])}{\partial a} =$$

$$\int_{y=b}^{\infty} m_1([a,y]) dy \int_{x=-\infty}^{a} m_2([x,b]) dx + \int_{x=-\infty}^{a} m_1([x,b]) dx \int_{y=b}^{\infty} m_2([a,y]) dy$$

which are the first two terms of definition 4.4. □

Theorem 4.9 *The conjunctive combination rule is associative, commutative, and if $m_1^{\mathcal{I}}$ is a vacuous belief function, then for may bbd $m_2^{\mathcal{I}}$, one has $m_{1 \circledcap 2}^{\mathcal{I}} = m_2^{\mathcal{I}}$.*

Proof. Immediate from relation (20). □

The result of the conjunctive combination rule can be neatly represented as:

Theorem 4.10 *We have:*

$$m_{1 \circledcap 2}^{\mathcal{I}}([a,b]) = \int_{x=-\infty}^{x=\infty} \int_{y=x}^{y=\infty} m_2^{\mathcal{I}}[[x,y]]([a,b]) \; m_1^{\mathcal{I}}([x,y]) dy dx \tag{21}$$

$$bel_{1 \circledcap 2}^{\mathcal{I}}([a,b]) = \int_{x=-\infty}^{x=\infty} \int_{y=x}^{y=\infty} bel_2^{\mathcal{I}}[[x,y]]([a,b]) \; m_1^{\mathcal{I}}([x,y]) dy dx \tag{22}$$

$$pl_{1 \circledcap 2}^{\mathcal{I}}([a,b]) = \int_{x=-\infty}^{x=\infty} \int_{y=x}^{y=\infty} pl_2^{\mathcal{I}}[[x,y]]([a,b]) \; m_1^{\mathcal{I}}([x,y]) dy dx \tag{23}$$

$$q_{1 \circledcap 2}^{\mathcal{I}}([a,b]) = \int_{x=-\infty}^{x=\infty} \int_{y=x}^{y=\infty} q_2^{\mathcal{I}}[[x,y]]([a,b]) \; m_1^{\mathcal{I}}([x,y]) dy dx \tag{24}$$

Proof. We prove the q relation, and from it the m relation. Other proofs are analogous. They all result from the fact that all transformations are linear ones. By theorem 4.1, one has $q_2^{\mathcal{I}}([x,y])(a,b) = q_2^{\mathcal{I}}([a,b])I_{[a,b]}^{[x,y]}$. From relations (7) and (20) we have:

$$\begin{aligned}
q_{1 \bigcirc\!\!\!\cap 2}^{\mathcal{I}}([a,b]) &= q_2^{\mathcal{I}}([a,b]) \iint_{x,y} m_1^{\mathcal{I}}([x,y])H(y-x)I_{[a,b]}^{[x,y]}dydx \\
&= \iint_{x,y} q_2^{\mathcal{I}}([a,b])I_{[a,b]}^{[x,y]} m_1^{\mathcal{I}}([x,y])H(y-x)dydx \\
&= \iint_{x,y} q_2^{\mathcal{I}}[[x,y]]([a,b]) m_1^{\mathcal{I}}([x,y])H(y-x)dydx \\
&= \int_{x=-\infty}^{x=\infty} \int_{y=x}^{y=\infty} q_2^{\mathcal{I}}[[x,y]]([a,b]) m_1^{\mathcal{I}}([x,y])dydx.
\end{aligned}$$

thus relation (24). Taking partial derivatives of both sides on a and b, one gets relation (21) for the bbd. □

4.3.2 Bayesian belief functions

We can deduce some practical properties dealing with Bayesian belief functions.

Theorem 4.11 *Suppose two bbds $m_1^{\mathcal{I}}$ and $m_2^{\mathcal{I}}$, and their conjunctive combination $m_{1 \bigcirc\!\!\!\cap 2}^{\mathcal{I}}$.*

1. *If any of them is a Bayesian belief function, then their normalized conjunctive combination is a Bayesian belief function.*
2. *If both of them are Bayesian belief function, then their normalized conjunctive combination is a Bayesian belief function.*

Proof. By definition 3.9, Bayesian belief functions can be represented as $g^{\mathcal{U}}(u,v) = \delta(u)f(v)$. As their only focal elements are the points on line with $u = 0$, so are the results of the intersection with any of the focal elements of any other bba. Therefore the focal elements of the result of the conjunctive combination are points with $u = 0$. This bbd is a Bayesian belief function provided the resulting bbd is normalized. When both bbds are Bayesian, the same proof still holds. □

4.3.3 Combining bbds and bbas

Bbds can also be defined on mixture of continuous and discrete spaces.

Theorem 4.12 *Suppose two bbds $m_1^{\mathcal{I}}$ and m_2^{Ω}, the first being on the $\mathcal{I}$ domain whereas the second is defined on a finite frame of discernment Ω. Their conjunctive combination $m_{1 \bigcirc\!\!\!\cap 2}^{\mathcal{I}\times\Omega}$ is given for every $[a,b] \in \mathcal{I}$ and $A \subseteq \Omega$ by:*

$$m_{1 \bigcirc\!\!\!\cap 2}^{\mathcal{I}\times\Omega}(([a,b],A)) = m_1^{\mathcal{I}}([a,b])m_2^{\Omega}(A).$$

Proof. Derived from vacuously extending both bbds on $\mathcal{I} \times \Omega$, and applying the conjunctive combination rule. $\square$

4.3.4 Conjunctive combination rule and specialization

Furthermore, the conjunctive combination rule can be represented as a specialization and the result of the combination is a specialization of its two components.

Theorem 4.13 *In relation (21), the function $m_2^{\mathcal{I}}[[x,y]]([a,b])$, the result of the conditioning of $m_2^{\mathcal{I}}$ on $[x,y]$ using Dempster conditioning rule (section 4.2), is a specialization function.*

Proof. To be a specialization, $m_2^{\mathcal{I}}[[x,y]]([a,b])$ must satisfy:

1. zero values constraint: relation (15). In theorem 4.2, the only positive terms for $m_2^{\mathcal{I}}[[x,y]]([a,b])$ are those for which $[a,b] \subseteq [c,d]$ and $\emptyset$. Hence $m_2^{\mathcal{I}}[[x,y]]([a,b]) = 0$ whenever $[a,b] \not\subseteq [x,y]$.
2. integral constraint: relation (16). This constraint is satisfied if $b_2^{\mathcal{I}}[[x,y]]([x,y]) = 1$. From theorem 4.3 with $[x,y] = [c,d]$ and $b = bel + m(\emptyset)$, one gets $b_2^{\mathcal{I}}[[x,y]]([x,y]) = 1 - bel_2^{\mathcal{I}}[[x,y]]((y,\infty]) - bel_2^{\mathcal{I}}[[x,y]]([-\infty,x))$. The two last terms are null as $m_2^{\mathcal{I}}[[x,y]]([a,b]) = 0$ whenever $[a,b] \not\subseteq [x,y]$.

Hence both constraints are satisfied. $\square$

Theorem 4.14 *$m_{1 \bigcirc\!\!\!\!\cap\, 2}^{\mathcal{I}}([a,b])$ is a specialization of both $m_1^{\mathcal{I}}$ and $m_2^{\mathcal{I}}$.*

Proof. That $m_{1 \bigcirc\!\!\!\!\cap\, 2}^{\mathcal{I}}$ is a specialization of $m_2^{\mathcal{I}}$ is deduced from theorem 4.13 and relation (21). As the conjunctive combination rule is symmetrical, the relation can be rewritten by interchanging the indexes 1 and 2. Therefore $m_{1 \bigcirc\!\!\!\!\cap\, 2}^{\mathcal{I}}$ is a specialization of $m_1^{\mathcal{I}}$. $\square$

We can also prove that the conjunctive combination rule is the only associative and commutative combination rule such that its result is a specialization of its two components and that commutes with the conditioning specialization (see theorem 4.6). This provides the major justification for its use within the TBM (Klawonn & Smets, 1992). But it concerns the justification of the conjunctive combination rule, what lays outside the scope of this paper.

5 Characteristic functions

5.1 Credal variables

Practically speaking, just as a random variable is a variable on the reals on which a pdf is defined, so a credal variable is a variable on the reals on which a bbd is defined. We present a more formal definition.

Definition 5.1 Credal space. *A credal space is a triple $(\Omega, \mathcal{A}, m^{\Omega})$ where Ω is a set, $\mathcal{A}$ a sigma-algebra defined on Ω and m^{Ω} is a bbd defined on $\mathcal{A}$.*

Definition 5.2 Credal variable. *A credal variable is a mapping from a credal space* $(\Omega, \mathcal{A}, m^{\Omega})$ *into* $\mathcal{R}$.

More subtle and more general definitions could be provided, but for our purpose, all we need is $\Omega = \mathcal{R}$, A being its Borel sigma-algebra (see section 3.4), $m^{\mathcal{I}}$ being a bbd which focal sets are closed intervals of $\mathcal{R}$ and the mapping being continuous.

5.2 The characteristic function related to $m^{\mathcal{I}}$

Just as characteristic functions are defined for probability density function (Kendall & Stuart, 1977, chapter 4), they can be extended directly to $m^{\mathcal{I}}$ thanks to the relation between $m^{\mathcal{I}}$ and $f^{\mathcal{I}}$:

Definition 5.3 Characteristic functions *The characteristic function of* $m^{\mathcal{I}}$ *is given by:*

$$\phi(t_1, t_2) = \iint\limits_{x,y} m^{\mathcal{I}}([x, y]) e^{it_1x+it_2y}\ dy\ dx.$$

Characteristic functions built from the commonality function and from the bbd are strongly related. The transform of the commonality function is the transform of the bbd divided by t_1t_2.

Theorem 5.1 *Let* $\phi(t_1, t_2)$ *be the characteristic function of* $m^{\mathcal{I}}$*, and* $\psi(t_1, t_2)$ *be the characteristic function of* $q^{\mathcal{I}}$ *given by:*

$$\psi(t_1, t_2) = \iint\limits_{x,y} q^{\mathcal{I}}([x, y]) e^{it_1x+it_2y}\ dy\ dx.$$

Then $\psi(t_1, t_2) = \phi(t_1, t_2)/(t_1t_2)$

Proof. Let G be a function on $\mathcal{R}^2$. Let $\Phi(t_1, t_2)$ be its characteristic function, which is a form of Fourier transform:

$$\Phi(t_1, t_2) = \iint\limits_{x,y} G(x, y) e^{it_1x+it_2y} dydx$$

Then

$$\phi(t_1, t_2) = \iint\limits_{x,y} \frac{\delta^2 G(x, y)}{\delta x \delta y} e^{it_1x+it_2y} dydx$$

satisfies:

$$\phi(t_1, t_2) = -t_1t_2\Phi(t_1, t_2).$$

Replacing $G(x, y)$ by $q^{\mathcal{I}}([x, y])$, and $\frac{\delta^2 G(x,y)}{\delta x \delta y}$ by $-m^{\mathcal{I}}([x, y])$ (see relation (12)), the transform of the commonality function is given by:

$$\iint\limits_{x,y} q^{\mathcal{I}}([x, y]) e^{it_1x+it_2y} dydx = \frac{1}{t_1t_2} \iint\limits_{x,y} m^{\mathcal{I}}([x, y]) e^{it_1x+it_2y} dydx$$

□

5.3 Adding credal variables

Suppose two credal variables. We can add them. As one could expect it, the characteristic function of their sum is the product of the individual characteristic functions, a widely used relation is statistics.

Theorem 5.2 *Let X_1 and X_2 be two credal variables defined on $\mathcal{I}$ which related probability density functions are given by $f_1^{\mathcal{I}}$ and $f_2^{\mathcal{I}}$. Let $\phi_i(t_1, t_2)$ be the characteristic functions of $f_i^{\mathcal{I}}$. Let the credal variable $Y = X_1 + X_2$. The characteristic function of its probability density function is given by:*

$$\phi_Y(t_1, t_2) = \phi_1(t_1, t_2)\phi_2(t_1, t_2), \text{ for all } (t_1, t_2).$$

Proof. Identical to the one used in probability theory. □

Example 3. Gauss-Gamma bbd Suppose several credal variable $X_1, ..., X_n$, each one being U-non-interactive Gauss-Gamma, with parameters (μ, σ, ν), thus $f(v : \mu, \sigma) = N(v : \mu, \sigma)$ and $h(u : \nu) = u^{\nu-1}e^{-u}/\Gamma(\nu)$ (see Example 2). The corresponding characteristic function is given by:

$$\phi(t_1, t_2) = e^{i\mu t_1 - \sigma^2 t_1^2/2}(1 - it_2)^{-\nu}.$$

The characteristic function of $Y = \sum_{j=1,...,n} X_j$ is given by:

$$e^{in\mu t_1 - n\sigma^2 t_1^2/2}(1 - it_2)^{-n\nu}.$$

So Y is also a Gauss-Gamma bbd with parameters $(n\mu, \sqrt{n}\sigma, n\nu)$.

6 Decision Making

6.1 Deriving $Betf$ from $f^{\mathcal{I}}$

Suppose a bbd $m^{\mathcal{I}}$ and its related density $f^{\mathcal{I}}$ where $f^{\mathcal{I}}$ is normalized. Let Bet be the pignistic transformation operator, hence $BetP = Bet(m^{\mathcal{I}}, \mathcal{R})$ where $BetP$ is the pignistic probability function and $\mathcal{R}$ is the betting frame (Smets, 2002, 2005). $BetP$ is defined for any X in the Borel sigma-algebra generated by $\mathcal{I}$. We define $BetF(a) = BetP([-\infty, a])$ and $Betf(a) = dBetF(a)/da$ as the pignistic distribution function and the pignistic density function, respectively.

The relation for the pignistic probability function becomes for $a < b$:

$$\begin{aligned} BetP([a,b]) &= \int_{x=-\infty}^{x=\infty} \int_{y=x}^{y=\infty} \frac{|[a,b] \cap [x,y]|}{|[x,y]|} f^{\mathcal{I}}(x,y) dy dx \\ &= \int_{x=-\infty}^{x=b} \int_{y=a \vee x}^{y=\infty} \frac{y \wedge b - x \vee a}{y - x} f^{\mathcal{I}}(x,y) dy dx \end{aligned}$$

where $|\emptyset|/|[x,y]| = 0$ and when $a < x = y < b$, the ratio $(y \wedge b - x \vee a)/(y - x) = 1$ by continuity. The next theorem provides the relation for $Betf$, the density function associated with $BetP$ with:

$$BetP([a,b]) = \int_{x=a}^{x=b} Betf(x) dx.$$

Theorem 6.1 *Given a bbd $m^{\mathcal{I}}$ and its related $f^{\mathcal{T}}$,*

$$Betf(a) = \lim_{\varepsilon \to 0} \int_{x=-\infty}^{x=a} \int_{y=a+\varepsilon}^{y=\infty} \frac{1}{y-x} f^{\mathcal{T}}(x,y) dy dx. \tag{25}$$

Proof. Let $b = a + \varepsilon$ where ε is a small positive real in the 0 neighborhood. Let $O(\varepsilon^n)$ denote any term of order n in ε, thus so that $\lim_{\varepsilon \to 0} O(\varepsilon^n)/n^k = 0$ whenever $k < n$. We get:

$$BetP([a, a+\varepsilon]) = \int_a^{a+\varepsilon} Betf(x)dx = Betf(a)\varepsilon + O(\varepsilon^2)$$

and

$$\begin{aligned}
BetP([a, a+\varepsilon]) &= \int_{x=-\infty}^{x=a+\varepsilon} \int_{y=a}^{y=\infty} \frac{y \wedge (a+\varepsilon) - x \vee a}{y-x} f^{\mathcal{T}}(x,y) dy dx \\
&= \int_{x=-\infty}^{x=a} \int_{y=a+\varepsilon}^{y=\infty} \frac{a+\varepsilon-a}{y-x} f^{\mathcal{T}}(x,y) dy dx \\
&\quad + \int_{x=a}^{x=a+\varepsilon} \int_{y=a+\varepsilon}^{y=\infty} \frac{a+\varepsilon-x}{y-x} f^{\mathcal{T}}(x,y) dy dx \\
&\quad + \int_{x=-\infty}^{x=a} \int_{y=a}^{y=a+\varepsilon} \frac{y-a}{y-x} f^{\mathcal{T}}(x,y) dy dx \\
&\quad + \int_{x=a}^{x=a+\varepsilon} \int_{y=a}^{y=a+\varepsilon} \frac{y-x}{y-x} f^{\mathcal{T}}(x,y) dy dx \\
&= \int_{x=-\infty}^{x=a} \int_{y=a+\varepsilon}^{y=\infty} \frac{\varepsilon}{y-x} f^{\mathcal{T}}(x,y) dy dx \\
&\quad + \varepsilon \int_{y=a+\varepsilon}^{y=\infty} \frac{a+\varepsilon-a}{y-a} f^{\mathcal{T}}(a,y) dy \\
&\quad + \varepsilon \int_{x=-\infty}^{x=a} \frac{a-a}{a-x} f^{\mathcal{T}}(x,a) dx \\
&\quad + \varepsilon^2 f^{\mathcal{T}}(a,a) + O(\varepsilon^2) \\
&= \varepsilon \int_{x=-\infty}^{x=a} \int_{y=a+\varepsilon}^{y=\infty} \frac{1}{y-x} f^{\mathcal{T}}(x,y) dy dx + O(\varepsilon^2)
\end{aligned}$$

For $\varepsilon \to 0$, we can write :

$$Betf(a) = \lim_{\varepsilon \to 0} \int_{x=-\infty}^{x=a} \int_{y=a+\varepsilon}^{y=\infty} \frac{1}{y-x} f^{\mathcal{T}}(x,y) dy dx$$

□

To use equation (25) of theorem 6.1, beware not to put directly $\varepsilon = 0$ as the term $1/(y-x)$ is undefined when $y = x$, which does not not occur in the correct integration (and explains why we went through these tedious derivations).

Theorem 6.2 *Given a bbd $m^{\mathcal{I}}$ and its related $g^{\mathcal{U}}$,*

$$Betf(a) = \lim_{\varepsilon \to 0} \int_{u=\varepsilon}^{u=\infty} \int_{v=a-u}^{v=a+u} \frac{1}{2u} g^{\mathcal{U}}(u,v) dv du. \tag{26}$$

6.2 Example: $Betf$ induced by a uniform density on $\mathcal{T}_{[0,1]}$.

Uniform density on $\mathcal{T}_{[0,1]}$ is achieved when $f^{\mathcal{T}}(x,y) = 2$ for all $x, y \in [0,1], x \leq y$. Then

$$\begin{aligned}
Betf(a) &= 2 \lim_{\varepsilon \to 0} \int_{x=0}^{x=a} \int_{y=a+\varepsilon}^{y=1} \frac{1}{y-x} dy dx \\
&= 2 \lim_{\varepsilon \to 0} \int_{x=0}^{x=a} log(y-x)|_{y=a+\varepsilon}^{y=1} dx \\
&= 2 \lim_{\varepsilon \to 0} \int_{x=0}^{x=a} (log(1-x) - log(a+\varepsilon-x)) dx \\
&= 2 \lim_{\varepsilon \to 0} (-(1-x)log(1-x) - x + ((a+\varepsilon-x)log(a+\varepsilon-x) + x)|_{x=0}^{x=a} \\
&= 2 \lim_{\varepsilon \to 0} (-(1-a)log(1-a) - a + \varepsilon log(\varepsilon) + a + 0 + 0 - (a+\varepsilon)log(a+\varepsilon) - 0) \\
&= -2((1-a)log(1-a) + alog(a))
\end{aligned}$$

7 Beliefs induced by a pdf

Suppose You collect a pdf on the set of real numbers $\mathcal{R}$. This pdf can represent two kinds of information.

7.1 The Bayesian belief function

In the first case, the pdf is understood as representing the agent's beliefs themselves. The result is a Bayesian belief function. It fits sometimes with objective data. Suppose a sensor which generated data x is corrupted by noise ε, so the collected data y is given by $y = x + \varepsilon$. Suppose the noise ε is generated by a random process with density h. Then Your belief about the value x generated by the sensor before corruption given the collected corrupted data y is also represented by a pdf fully determined by h. Of course other cases can be considered that result in a Bayesian belief function.

Let $h(v)$ be the collected pdf defined on $\mathcal{R}$ and let P be the probability measure related to h: $P([a,b]) = \int_{v=a}^{v=b} h(v) dv$. (We use the (u,v) notation as is is simpler.) Then $g^{\mathcal{U}}(u,v) = \delta(u)\lambda(u,v)$ with:

$$\begin{aligned}
\lambda(u,v) &= h(v) \text{ if } u = 0 \\
&= 0 \quad \text{otherwise.}
\end{aligned}$$

We prove that $bel^{\mathcal{I}}([a,b]) = P([a,b])$ as it should.

Theorem 7.1 *Suppose* $m^{\mathcal{I}}([a,b]) = \delta(u)\lambda(u,v)$ *with* $\lambda(u,v) = h(v)$ *if* $u = 0$ *and 0 otherwise, where h(v) is a pdf on* $\mathcal{R}$*. Let* $P([a,b]) = \int_{t=a}^{t=b} h(t)dt$ *be the probability that the random variable which pdf is* $h(t)$ *is in* $[a,b]$*. Then*

$$bel^{\mathcal{I}}([a,b]) = P([a,b]).$$

Proof. Given the relations of section 2.1, we have:

$$bel^{\mathcal{I}}([a,b]) = bel^{\mathcal{I}}([v-u, v+u]) = \int_{z=0}^{z=u} \int_{t=v-u+z}^{t=v+u-z} g^{\mathcal{U}}(z,t)dtdz$$

$$= \int_{z=0}^{z=u} \delta(z) \int_{t=v-u+z}^{t=v+u-z} \lambda(z,t)dtdz$$

$$= \int_{t=v-u}^{t=v+u} \lambda(0,t)dt = \int_{t=v-u}^{t=v+u} h(t)dt = P([a,b])$$

□

7.2 LC bbd induced by Betf

In the second case, one considers that the collected pdf represents how the agent would bet about the actual value of the unknown variable defined on the frame $\mathcal{R}$. Thus the pdf is the pignistic probability function $Betf$ induced on $\mathcal{R}$ by the underlying belief function which value is unknown.

Many $m^{\mathcal{I}}$ functions can induce this $Betf$ function. The set of bba $m^{\mathcal{I}}$ which related pignistic probability density function equals $Betf$ is called the set of isopignistic belief functions induced by $Betf$ and denoted $\mathcal{B}Iso(Betf)$. So if Bet is the operator that corresponds to the pignistic transformation, i.e., $Betf = Bet(m)$, then $\mathcal{B}Iso(Betf) = Bet^{-}(Betf)$ is the (maybe generalized) inverse image of $Betf$ by Bet.

The user knows only that $bel^{\mathcal{I}} \in \mathcal{B}Iso(Betf)$. The least commitment principle (never give more belief than justified) can be evoked to select the least committed belief function in $\mathcal{B}Iso(Betf)$.

We analyze two cases. In the first we do not know $Betf(x)$ for every $x \in \mathcal{R}$, but only a finite (or at most countable) numbers of values $x_i : i = 1, 2, ...$ In the second case $Betf(x)$ is known for every $x \in \mathcal{R}$.

7.2.1 $BetF$ known for some $x \in \mathcal{R}$

In many practical applications, the values of $BetF$ are assessed only for a few x values. One can then try to determine the missing values, using some assumed underlying parametric model. In that case, we are back to the case treated in the next section 7.2.2. Another approach consists in using what is available and fitting the least committed bbd which pignistic transformation satisfies to the given constraints. The resulting bbd is made of a finite number of masses (formally, weighted Dirac's functions). Their focal elements are not always intervals. We will only consider the case of interval focal elements, even though this solution can easily be adapted to handle the general case. We present an example related to reliability study. We then explain the algorithm to build the q-LC bbd isopignistic with the given $BetF$ values. We present the value of $Betf$ induced by this bbd. We then present the concept of expectation. Finally we present an example related to reliability studies where discounting and conjunctive combination rule are also applied.

Example 4. To assess the reliability of an equipment, one needs the value of π, the probability that the equipment fails within a given time. The parameter π is often not available. It is assessed by experts who provide their opinion about the value of π. For instance, they are asked to express a value $p_{.50}$ such they believe as much that the actual value π is below $p_{.50}$ or above $p_{.50}$. Then they are asked for a value $p_{.25}$ such that they are three times more confident that π is larger then $p_{.25}$ than smaller. In fact they produce some of the percentiles of a meta-probability function about the value of the probability π. The percentiles classically collected and published are the .05, .25, .50, .75, .95 percentiles, or some subset of them.

A percentile like the .05 percentile is the value $p_{.05}$ such that the expert is ready to bet that $\pi \leq p_{.05}$ versus $\pi \geq p_{.05}$ with odds 5 to 95. These are just our pignistic probabilities. Percentile $p_{.05}$ satisfies $BetP(\pi \leq p_{.05}) = .05$, and similarly for the other percentiles. So in general $BetP(\pi \leq p_x) = x$ for $x \in [0, 1]$.

Suppose for simplicity sake that we have only collected the .05, .50 and .95 percentiles. Assume the collected percentiles are: $p_{.05} = .5, p_{.50} = .7, p_{.95} = .8$. This is all we know about the expert beliefs about the value of π. They are the pignistic probabilities induced by an underlying belief function defined on $[0, 1]$. There are many such belief functions. The Minimal Commitment Principle can be evoked. Finding the q-least committed belief function which pignistic transformation satisfies the known constraints $BetP$ is computationally trivial.

The bbm given to the whole interval $[0, 1]$ must satisfy $BetP([0, .5]) = .05$. The bbm given to [0,1] is spread equally on the interval [0,1] by the pignistic transformation. The value m([0,1]) = .1 explains the .05 given to [0, .5] and is compatible with the two other data. The next constraint to be satisfied is the $p_{.95} = .8$. The [.8, 1] interval received already a probability of $m([0, 1]) \times (1 - .8) = .1 \times .2 = .02$. The bbm that could justify the still unexplained pignistic probability $0.05 - 0.02 = 0.03$ to be allocated to $[.8, 1]$ is to be given to the largest left over interval, i.e. $[.5, 1]$. The portion of that bbm given to $[.8, 1]$ - i.e. $(1-.8)/(1-.5) = 2/5$ - must be equal to 0.03. Hence $m([.5, 1.]) = .03*5/2 = .075$.

The next bbm are computed similarly. The results are $m([.0, 1.]) = .100$, $m([.5, 1.]) = .075$, $m([.5, .8]) = .600$, $m([.7, .8]) = .225$. Table 3 presents the whole computation. We start from the known $BetP$. We created the largest focal element and give it the largest mass compatible with the $BetP$. We subtract the masses so allocated, and repeat the operation on the residuals.

In general the solution is given by the next theorem.

Theorem 7.2 *Let Ω be a subset of $\mathcal{R}$. Let $\{\omega_0, \omega_1, ..., \omega_n\}$ be a set of elements in Ω with $\omega_i \leq \omega_{i+1}$ and $\Omega = [\omega_0, \omega_n]$. Let BetP be known on the intervals $[\omega_i, \omega_{i+1}]$. Let m be the q-LC bba isopignistic with BetP.*

Let Betf be the pdf on Ω such that $BetP([\omega_{i-1}, \omega_i]) = \int_{x=\omega_{i-1}}^{x=\omega_i} Betf(x)dx$ and $Betf(x)$ is constant for all $x \in [\omega_{i-1}, \omega_i]$. Then

$$Betf(x) = BetP([\omega_{i-1}, \omega_i])/([\omega_i - \omega_{i-1}]), x \in [\omega_{i-1}, \omega_i]$$

k	N_k	X_k	I	limits → width →	0.-.5 .5	.5-.7 .2	.7-.8 .1	.8-1. .2
1	1,2,3,4	[.0,1.]	1	$R_i^1 = BetP$	.050	.450	.450	.050
				$m = .100$	.050	.020	.010	.020
2	2,3,4	[.5,1.]	4	residual R_i^2	.000	.430	.440	.030
				$m = .075$		.030	.015	.030
3	2,3	[.5,.8]	2	residual R_i^3	.000	.400	.425	.000
				$m = .600$		.400	.200	
4	3	[.7,.8]	3	residual R_i^4	.000	.000	.225	.000
				$m = .225$			.225	
5	$\emptyset$			residual R_i^5	.000	.000	.000	.000

Table 3: Building the q-LC Isopignistic bba for the reliability assessment based on expert opinions, using the algorithm of theorem 7.2.

Then the next algorithm builds m.

$N_1 = [1, ...n]$

$R_i^1 = BetP([\omega_{i-1}, \omega_i]), \forall i \in N_1$

$k = 1$

While $N_k \neq \emptyset$

$X_k = \cup_{\nu \in N_k} [\omega_{\nu-1}, \omega_\nu]$

Find $I \subseteq N_k : \dfrac{R_i^k}{\omega_i - \omega_{i-1}} = \min_{j \in N_k} \{ \dfrac{R_j^k}{\omega_j - \omega_{j-1}} \} \ \forall i \in I$

Let: $m(X_k) = \dfrac{R_i^k}{\omega_i - \omega_{i-1}} \sum_{j \in N_k} (\omega_j - \omega_{j-1})$

$R_j^{k+1} = R_j^k - \dfrac{m(X_k)(\omega_j - \omega_{j-1})}{\sum_{j \in N_k} (\omega_j - \omega_{j-1})}, \forall j \in N_k$

$N_{k+1} = N_k \backslash I$

$k = k + 1$

End while

If $Betf$ is bell shaped[7]*, then the focal elements of the q-LC bba isopignistic with $BetP$ are nested intervals. If $Betf$ is not bell shaped, then some focal elements of the q-LC bba isopignistic with $BetP$ are the union of more than one interval.*

The next theorem presents the value of the pdf generated by the bbd build in theorem 7.2.

Theorem 7.3 (Probability density function) *Suppose a bbd m on $\mathcal{R}$ made of masses given to the intervals $[a_i, b_i], i = 1, 2, ...$ The related pdf $f(x) : x \in \mathcal{R}$, is given by*

$$f(x) = \sum_i m([a_i, b_i]) I(x, [a_i, b_i])/(b_i - a_i)$$

[7] A 'bell shaped' density is a unimodal density, continuous and strictly monotone increasing (decreasing) at left (right) of the mode.

where $I(x, [a, b]) = 1$ if $x \in [a, b]$ and 0 otherwise.

Proof. Each mass $m([a_i, b_i])$ is equally distributed on its focal element, so its density at x is $1/(b_i - a_i)$ if $x \in [a_i, b_i]$, and 0 otherwise, hence the I coefficient. The value of $f(x)$ is obtained by adding the densities produced by each mass. □

Expectations are taken using $Betf$.

Theorem 7.4 (Expectation) *Suppose the pdf $f(x)$ defined in theorem 7.3. The expectation of the function $g : \mathcal{R} \to \mathcal{R}$ is given by:*

$$E(g) = \sum_i \frac{m([a_i, b_i])}{b_i - a_i} \int_{a_i}^{b_i} g(x)dx$$

Proof. One has

$$\begin{aligned} E(g) &= \int_{-\infty}^{\infty} g(x)f(x)dx \\ &= \int_{-\infty}^{\infty} g(x) \sum_i \frac{m([a_i, b_i])}{b_i - a_i} I(x, [a_i, b_i])dx \\ &= \sum_i \frac{m([a_i, b_i])}{b_i - a_i} \int_{a_i}^{b_i} g(x)dx \end{aligned}$$

□

In particular, the mean (where $g(x) = x$) is computed as:

$$\text{mean } = \sum_i m([a_i, b_i])(a_i + b_i)/2$$

Ω	m_1	$m_1^{.7}$	Ω	m_2	$m_2^{.6}$
.0-1.	0.100	0.3700	.0-1.	0.25	0.55
.5-1	0.075	0.0525	.4-1.	0.05	0.03
.5-.8	0.600	0.4200	.4-.7	0.50	0.30
.7-.8	0.225	0.1575	.6-.7	0.20	0.12
sum	1	1	sum	1	1
mean	0.665	0.6155	mean	0.565	0.539

Table 4: Building the q-LC Isopignistic bba for the reliability assessment based on the opinions of the two experts, and their discounted bba.

Example 5. Consider the same problem as in the previous example, but we collect data from two experts. The percentiles of their meta-probability about the value of π are given, respectively,by:

1. Expert 1: $p_{.05} = .5, p_{.50} = .7, p_{.95} = .8$

2. Expert 2: $p_{.10} = .4, p_{.50} = .6, p_{.90} = .7$.

In table 4, we present the focal elements and the bbm of the q-LC bbd isopignistic with the collected percentiles (column Ω, m_1 and m_2).

I am collecting these two sets, and my own opinion about the two experts is limited. I feel that the bbas must be discounted by a factor .3 for expert 1, and .4 for expert 2. The results of discounting m_1 and m_2 are given in the columns $m_1^{.7}$ and $m_2^{.6}$. Data m_1 (m_2) are multiplied by .7 (.6) and the mass .3 (.4) is added to the universe ($[.0, 1.]$).

The means for the four bbds are presented in table 4.

The discounted bbd are combined by the conjunctive combination rule (section 4.3). The result of the combination are presented in table 5, columns 1 and 2.

The pdf $f(x)$ derived from m is also given in table 4. The top line presents the upper boundaries of the focal elements. The 1's in the table indicate the value of $I(x, [a, b])$ where b is the upper limit of the interval indicated at the top of the columns, and a is the value to its left (with 0 at left of 0.4). Beware that there is a Dirac's function with weight .06615 on .7. The pdf $f(x)$ is constant in each intervals. The width of the intervals are given on next line. The product width$\times f(x)$ is the integral of $f(x)$ over the corresponding interval. Finally $F(x)$ is the cumulative distribution function. The weight 0.06615 of the Dirac's centered at .7 is included in the $F(.7)$ value.

The mean computed from the pdf in table 4 is 0.61733. The $p_{.50} = 0.6446$ is obtained by solving the next interpolation:

$$.50 = .358 + (p_{.50} - .6) \times .3185/.1.$$

int	m	0.4	0.5	0.6	0.7	0.8	1
.0-1.	0.2035	1	1	1	1	1	1
.4-1.	0.0111		1	1	1	1	1
.4-.7	0.111		1	1	1		
.6-.7	0.1011				1		
.5-1.	0.03045			1	1	1	1
.5-.7	0.14175			1	1		
.5-.8	0.2436			1	1	1	
.7-.8	0.09135					1	
.7	0.06615						
	$f(x)$	0.2035	0.592	2.17365	3.18465	2.0084	0.2829
	width	0.4	0.1	0.1	0.1	0.1	0.2
	width$\times f(x)$	0.0814	0.0592	0.217365	0.318465	0.20084	0.05658
	$F(x)$	0.0814	0.1406	0.357965	0.74258	0.94342	1

Table 5: The bbd $m = m_1^{.7} \textcircled{\cap} m_2^{.6}$ and the pdf induced by m. Based on data of table 4.

7.2.2 Continuous pdf

Just as in the finite case, the q-LC element of $\mathcal{BI}so(Betf)$ is a consonant belief function (the proof for the finite case can be found in (Dubois, Prade, & Smets, 2003) and extend directly to the present case when the $Betf$ is a 'bell shaped' density. On $\mathcal{R}$, this will imply that the focal elements are nested.

In this paper, we focus only on the case of 'bell shaped' densities. The determination of the bbd on $\mathcal{R}$ described in the next theorem is based on focal elements $[a, b]$ whose limits a and b share the same density, thus $Betf(a) = Betf(b)$. $Betf$ being unimodal and strictly monotone increasing or decreasing, each of a and b uniquely determines the other one. Being bell shaped, $Betf(x) > Betf(a)$ for all $x \in (a, b)$. In the theorem we use the upper limit b, and given b, we define a as $\gamma(b)$. The determination of $\gamma(b)$ is simple if $Betf$ is an exponential distribution ($\gamma(b) = 0$ for all b), or if $Betf$ is a symmetrical unimodal distribution (like the Laplace and in the Gaussian distributions) ($\gamma(b)$ satisfies $\nu - \gamma(b) = b - \nu$ when $b \geq \nu$ and ν is the mode of the pdf).

In the next two theorems, we derive a consonant bba which belongs to $\mathcal{B}Iso(Betf)$, and then prove that his solution is the q-least committed element of $\mathcal{B}Iso(Betf)$.

Theorem 7.5 *Let $Betf$ be a 'bell shaped' pignistic probability function on $\mathcal{R}$ with mode ν. Let the bba $m^{\mathcal{I}}([a, b]) = \theta(b)\delta(a - \gamma(b))$ and let $\gamma(b)$ satisfy $Betf(\gamma(b)) = Betf(b), b \geq \nu$. Then:*

$$\theta(b) = (\gamma(b) - b)\frac{dBetf(b)}{db}. \tag{27}$$

This bba is a consonant bba and belongs to $\mathcal{B}Iso(Betf)$.

Proof. By theorem 3.7, the bba of the theorem is consonant. Its pignistic transformation is given for $b \geq \nu$ by:

$$Betf(b) = \int_{y=b}^{y=\infty} \theta(y)\frac{1}{y - \gamma(y)}dy.$$

Derivating both terms for b gives the theorem for $b \geq \nu$. The same derivation holds for $b \leq \nu$. Thus the theorem. □

Theorem 7.6 *Let $Betf$ be a 'bell shaped' pignistic probability function on $\mathcal{R}$ with mode ν. Let $\gamma(b)$ satisfies $Betf(\gamma(b)) = Betf(b), b \geq \nu$. The $\mathcal{U}-$form of the bba $m^{\mathcal{I}}([a, b])$ in theorem 7.5 is given by $g^{\mathcal{U}}(u, v) = h(u)\delta(v - \phi(u))$ where $u = (b - \gamma(b))/2, \phi(u) = (b + \gamma(b))/2$ Then:*

$$h(u) = 2(b - \gamma(b))\frac{f'(b)f'(\gamma(b))}{f'(b) - f'(\gamma(b))} \tag{28}$$

where $f'(x) = \frac{dBetf(\square)}{d\square}|_{\square=x}$.

Proof. We write $f(x)$ for $Betf(x)$. Fix $b > \nu$, determine $\gamma(b)$ and define $u = (b - \gamma(b))/2$. We have:

$$pl(b) = \int_u^\infty h(x)dx = f(b)(b - \gamma(b)) + \int_{-\infty}^{\gamma(b)} f(x)dx + \int_b^\infty f(x)dx.$$

Derive both terms on b. One gets:

$$h(u)\frac{1 - \gamma'(b)}{2} = f'(b)(b - \gamma(b)) + Betf(b)(1 - \gamma'(b) + \gamma'(b) - 1).$$

From $Betf(b) = Betf(\gamma(b))$, one gets after derivating both terms on b, $f'(b) = f'(\gamma(b))\gamma'(b)$ where γ' is the derivate of γ.

By replacing $\gamma'(b)$ with $f'(b)/f'(\gamma(b))$, one gets the relation (28).

Similar results hold for $b < \nu$. At $b = \nu$, $b = \gamma(b)$, the derivatives are undefined, but by continuity, one gets $h(0) = 0$. □

The relation of theorem 7.6 is not very useful in practice. Starting with u is not sufficient. One must then find out the v value such that $Betf(v-u) = Betf(v+u)$, what is usually not easy to derive. Once u and v are known, one can use the relation of theorem 7.6, but as (u, v) uniquely determines b and $\gamma(b)$, the relations of theorem 7.5 seem easier to manipulate.

We now prove that the bba of theorem 7.5 is the q-least committed element of $\mathcal{B}Iso(Betf)$.

Theorem 7.7 *The bba $m^{\mathcal{I}}([a,b])$ of theorem 7.5 is the q-least committed element of $\mathcal{B}Iso(Betf)$.*

Proof. Suppose we have some bbd of non null measure located in the neighborhood of $(x, y) \in \mathcal{T}$ which does not belong to the focal elements of the consonant bbd given in theorem 7.5. We can then always find out a point $(x*, y*)$ which is a focal element of the consonant bbd given in theorem 7.5 and such that (x, y) belongs to either the lower left or the upper right quadrant centered on $(x*, y*)$.

In order to fit with the pignistic transformation constrain, it means that this masses around (x, y) must be taken away from those masses in the upper left quadrant.

But $q^{\mathcal{I}}([x*, y*])$ is the integral over the bbd in the upper left quadrant. The existence of some bbd (of non null measure) outside this quadrant means that the commonality at $(x*, y*)$ is smaller than with the solution of relation theorem 7.5. So whenever non null densities do not belong to this solution, $q^{\mathcal{I}}([x*, y*])$ is smaller. Thus the $q^{\mathcal{I}}([x*, y*])$ of the solution of relation theorem 7.5 are always the largest possible, and thus the solution is the q-LC solution. □

We prove a few useful theorems.

Theorem 7.8 *If $Betf$ is symmetrical, centered on ν, then $v(u)$ is the line perpendicular to the diagonal and crosses the diagonal at ν, the mode of $Betf$.*

Proof. As $Betf$ is symmetrical, $\gamma(b) = \nu - b$, and ν is the midpoint of every focal element. Hence the theorem. □

Theorem 7.9 *Let ν be the mode of $Betf$. Then $pl^{\mathcal{I}}([\nu, \nu]) = 1$.*

Proof. All focal elements contain ν, hence the theorem. □

Theorem 7.10 *The bbd $m^{\mathcal{I}}$ of theorem 7.7 is normalized.*

Proof. Immediate as $pl^{\mathcal{I}}([\nu, \nu]) = 1$ from theorem 7.9. Thus $pl^{\mathcal{I}}(\mathcal{I}) = 1$. □

Example 6. The Gaussian $Betf$. Suppose $Betf$ is a gaussian distribution $N(x : \mu, \sigma)$ with mean μ and standard deviation σ. Let $x \geq \mu$. One has $\gamma(x) = 2\mu - x$. Hence $-(x - \gamma(x)) = 2(\mu - x)$. We have:

$$\begin{aligned}\theta(x) &= 2(\mu - x)\frac{d}{dx}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} \\ &= -2(\mu - x)\frac{1}{\sqrt{2\pi}\sigma}\frac{x-\mu}{\sigma^2}e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} \\ &= 2(x-\mu)^2\frac{1}{\sqrt{2\pi}\sigma^3}e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}\end{aligned}$$

This function is 0 at $x = \mu$, increases with x and reaches a maximum of $4/(\sigma e\sqrt{2\pi})$ at $x = \mu + \sqrt{2}\sigma$ (obtained from $d\theta(x)/dx = 0$), then decreases to 0 at x goes to infinity.

8 The General Bayesian Theorem

To apply the General Bayesian Theorem (Smets, 1993; Delmotte & Smets, 2004), all we need is the likelihood vector lkh on the finite set of hypotheses, and the likelihood $lkh(\theta_i)$ given to the hypothesis $\theta_i \in \Theta$ generated by the observation $X \subseteq \mathcal{R}$ is equal to the plausibility of observing X if the actual hypothesis were θ_i.

8.1 Isopignistic q-LC belief function induced by a pdf

For many applications, one can expect that the available data are the conditional pignistic density $Betf[\theta_i]$ for each $\theta_i \in \Theta$.

We transform these densities into their q-least committed isopignistic bbd $m^{\mathcal{I}}[\theta_i]$ as done in section 7.2. Then given the observed data $X \subseteq \mathcal{R}$, be it a point or an interval or some more complex type of data, we assess $pl^{\mathcal{I}}[\theta_i](X)$.

As $m^{\mathcal{I}}[\theta_i]$ is consonant, $pl^{\mathcal{I}}[\theta_i](X) = pl^{\mathcal{I}}[\theta_i](x)$ where $x = \min(x \in X)$ if $\min(x \in X) > \nu$, $x = \max(x \in X)$ if $\max(x \in X) < \nu$ and $x = \nu$ otherwise, where ν is the mode of the $Betf$ density.

Theorem 8.1 *Let $Betf$ be a 'bell shaped' pignistic density function with mode ν. Let $X = [x, y]$. Suppose $x > \nu$. Then*

$$pl^{\mathcal{I}}[\theta_i](X) = \int_{t=x}^{t=\infty}(\gamma(t) - t)\frac{dBetf(t)}{dt}dt, \tag{29}$$

where $\gamma(t)$ satisfies $Betf(\gamma(t)) = Betf(t)$. If $Betf$ is symmetrical, then

$$pl^{\mathcal{I}}[\theta_i](X) = 2(x - \nu)Betf(x) + 2\int_{t=x}^{t=\infty} Betf(t)dt. \tag{30}$$

Proof. All densities covering x cover X, and only them. Thus we must integrate relation (27) for $t \geq x$, hence relation (29). Using the relation $uv' = (uv)' - u'v$, with $u = \gamma(t) - t$ and $v = Betf(t)$, we get:

$$
\begin{aligned}
pl^{\mathcal{I}}[\theta_i](X) &= (\gamma(t) - t)Betf(t)|_x^{\infty} - \int_{t=x}^{t=\infty} (\frac{d\gamma(t)}{dt} - 1)Betf(t)dt \\
&= (x - \gamma(x))Betf(x) - \int_{t=x}^{t=\infty} (\frac{d\gamma(t)}{dt} - 1)Betf(t)dt.
\end{aligned}
$$

If furthermore, $Betf$ is symmetrical with mode ν, then $\gamma(x) = 2\nu - x$, and we get:

$$pl^{\mathcal{I}}[\theta_i](X) = 2(x - \nu)Betf(x) + \int_{t=x}^{t=\infty} 2Betf(t)dt.$$

□

The case where $\max(x \in X) < \nu$ is solved identically, and when $\nu \in X$, $pl^{\mathcal{I}}[\theta_i](X) = 1$ by theorem 7.9.

Relation (29) can be expressed in a quite different way initially described in (Dubois, Prade, & Sandri, 1993, relation (6)).

Theorem 8.2 *Let $Betf$ be a 'bell shaped' pignistic density function with mode ν. Let $X = [x, y]$. Suppose $x > \nu$. Then*

$$pl^{\mathcal{I}}[\theta_i](X) = \int_{t=-\infty}^{t=\infty} min(Betf(t), Betf(x))dt \tag{31}$$

Proof. Take the derivative of relations (29) and (31) on x. We get from (29):

$$-(\gamma(x) - x)\frac{dBetf(x)}{dx}$$

and from (31):

$$\int_{t=x}^{t=\gamma(x)} \frac{dBetf(x)}{dx}dt = (x - \gamma(x))\frac{dBetf(x)}{dx}.$$

The equality of the constant terms is handled by the fat both pdf are normalized.
□

The other cases where $\max(x \in X) \leq \nu$ are solved identically.

As $m^{\mathcal{I}}$ is consonant, $bel^{\mathcal{I}}$, $pl^{\mathcal{I}}$ and $q^{\mathcal{I}}$ satisfy useful relations.

Theorem 8.3 *Let $m^{\mathcal{I}}$ be the consonant bbd of theorem 7.5. Let ν be the mode of $Betf$. One has for all $[a, b] \subseteq \mathcal{I}$:*

$$
pl^{\mathcal{I}}([a, b]) = \begin{cases} pl^{\mathcal{I}}([a, a]) & \text{if } \nu \leq a \leq b \\ 1 & \text{if } a \leq \nu \leq b \\ pl^{\mathcal{I}}([b, b]) & \text{if } a \leq b \leq \nu \end{cases}
$$

$$q^{\mathcal{I}}([a, b]) = min(q^{\mathcal{I}}([a, a]), q^{\mathcal{I}}([b, b]))$$

with $q^{\mathcal{I}}([a, a]) = pl^{\mathcal{I}}([a, a]), \forall a \in \mathcal{I}$.

Proof. For $q^{\mathcal{I}}$, draw the right angle centered on $[a, b]$. Observe that $m^{\mathcal{I}}$ can enter into it only by the lower or the right side, and cannot leave the domain of $q^{\mathcal{I}}([a, b])$ as being consonant, the focus of the focal element propagates always in the upper left quadrant. If it enters by the bottom, $q^{\mathcal{I}}([a, b]) = q^{\mathcal{I}}([b, b])$ Similar reasoning are used for the other properties. □

As we can expect $Betf$ to be a classical probability density function, like a Gaussian, a Laplace, a gamma, etc..., the integrals are well documented and the programs to compute them are easily accessible.

Therefore, the General Bayesian Theorem can be extended to the case where the observation is defined on $\mathcal{R}$.

Example 7. A sensor with Gaussian observations. Suppose a sensor S that reports the likelihood on the set of hypotheses $\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4\}$. Suppose the measurement space X is $\mathcal{R}$ and we know $Betf^X[\theta_i]$ for each hypothesis. Let $Betf^X[\theta_i](x) = N(x; \mu_i, \sigma_i), x \in X$. The values of the parameters are presented in table 6.

I	μ_i	σ_i	$pl^X[\theta_I]([15, 35])$	$BetP^{\Theta}[15, 35]]$
1	10	4	0.5337	0.1409
2	20	8	1	0.3094
3	30	5	1	0.3094
4	40	10	0.8316	0.2402

Table 6: For each of the four hypothesis $\theta_i \in \Theta$, the parameters of the Gaussian distributions, the likelihoods (the plausibility of the observation given each hypothesis) and the posterior pignistic probabilities on Θ.

Suppose the observation is imprecise and given by the interval $[x_1, x_2] = [15, 35]$. The likelihoods are:

$$pl^{\mathcal{I}}[\theta_1]([x_1, x_2]) = 2(x_1 - \mu_1)N(x_1 : \mu_1, \sigma_1) + \int_{t=x_1}^{t=\infty} 2N(t : \mu_1, \sigma_1)dt = 0.5337$$

$$pl^{\mathcal{I}}[\theta_2]([x_1, x_2]) = 1$$

$$pl^{\mathcal{I}}[\theta_3]([x_1, x_2]) = 1$$

$$pl^{\mathcal{I}}[\theta_4]([x_1, x_2]) = 2(\mu_4 - x_2)N(x_2 : \mu_4, \sigma_4) + \int_{t=-\infty}^{t=x_2} 2N(t : \mu_4, \sigma_4)dt = 0.8316.$$

It is interesting to note that if $x_1 > \mu_i$ (respectively $x_2 < \mu_i$), the likelihoods do not depend on x_2 (respectively x_1). Furthermore if the mode is covered by the observation, the likelihood is 1. In table 6, we present the values of the parameters and the likelihoods, which are equal to $pl^X[\theta_i]([x_1, x_2])$.

From these likelihoods, one computes $m^{\Theta}[[x_1, x_2]]$ using the General Bayesian Theorem formulas and from this bba one computes $BetP^{\Theta}$ for each hypothesis. These $BetP^{\Theta}$ are presented in table 6. In practice, we use very efficient short cuts to transform likelihoods into $BetP^{\Theta}$. They can be found in the software TBMLAB[8] developed by Smets and Denoeux. In the present case, hypotheses θ_2 and θ_3 are the most supported hypotheses.

[8]Downloadable from http://iridia.ulb.ac.be/~psmets

It is worth noticing that whenever the observation $x \subseteq X$ covers the mode of the pdf on X, then $pl^X[\theta](x) = 1$ (see theorem 7.10). This means that every such hypothesis will get the same a posteriori $BetP^\Theta$, a situation not encountered in the Bayesian analysis where the posterior probability favors the hypothesis with the smallest variance. This property deserves some consideration (Ristic & Smets, 2004). In our example, θ_2 and θ_3 are such hypotheses. So under θ_2 and θ_3, the observed data cannot be more plausible, more likely than what we have observed, so there is no reason why one of them should be more supported than the other. The TBM will give them equal supports, contrary to the Bayesian analysis. This is still another property that might help to choose the 'good' model.

8.2 Point observations and Bayesian belief function

Suppose the conditional belief functions $f^X[\theta_i], \theta_i \in \Theta$, over the observation domain X are Bayesian as considered in section 7.1. The likelihood of hypothesis θ_i given the observation $A \subseteq X$ is equal to $pl^X[\theta_i](A) = \int_A f^X[\theta_i](x)dx$.

Suppose the observation is the point $x \in X$. The likelihood becomes $pl^X[\theta_i](x) = f^X[\theta_i](x)dx$, an infinitesimal. In that case the normalized posterior belief function on Θ is obtained by computing the classical GBT, but letting dx tend to 0. The result is given by :

$$pl^\Theta[x](\theta_i) = \frac{f^X[\theta_i](x)}{\sum_{\theta_j \in \Theta} f^X[\theta_j](x)}.$$

This posterior plausibility function is in fact a probability function that is often encountered in probability theory. This particular result comes from the infinite information provided by the observation x. Indeed claiming to have observed $x \in X$ implies an infinite precision as the data is known for all its decimals. Infinite information like 'observing x' is at the origin of the fact the posterior belief function is a probability function. This illustrates the position of classical Bayesian statistics in the TBM: Bayesian statistics result from infinitely precise data.

9 Conclusions

Classically, belief functions are defined on frames of finite cardinalities. In fact this limitation can be relaxed. Belief functions can be defined on $\mathcal{R}$, the set of real numbers, provided their focal elements can be defined by a finite number of parameters. In that case, one can define a probability density function (pdf) that plays the role of the basic belief masses. Masses become densities, and the belief functions, plausibility functions and commonality functions are integrals of this pdf.

In this paper, we consider belief functions on the set of reals, assuming the focal elements are the closed intervals of $\mathcal{R}$. The belief function and its related functions are defined on the Borel sigma-algebra generated by the closed interval of $\mathcal{R}$.

We present most of the relations encountered in the TBM in this new setting.

In many practical cases, one can expect that the only available information is a pignistic probability density function on $\mathcal{R}$. In that case, the least commitment principle can be invoked to justify the construction of the q-least committed isopignistic belief function induced by the given pignistic probability density function. The solution is a consonant belief function on $\mathcal{R}$ whose value is presented here.

We think these extensions of the TBM will be useful in many practical contexts, as already illustrated in the applications presented in (Ristic & Smets, 2004).

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