THE AXIOMATIC JUSTIFICATION OF THE TRANSFERABLE BELIEF MODEL.

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Summary: Belief functions have recently been advocated as an alternative to probability functions for representing quantified belief. This new normative model has several merits, but these merits are not sufficient to justify its use. Some 'axiomatic' justification is also needed. Indeed the examination of the requirements that underlie the normative models of subjective behaviors provides usually the best if not the only tool to compare them. We present such a set of axioms. In order to show that belief functions are appropriate for representing quantified beliefs, we present and analyze the requirements that should be satisfied when conditioning is introduced and when the domain on which beliefs are assessed changes. The deduced model corresponds to the transferable belief model, i.e. a model for quantified beliefs based on belief functions and independent of any underlying probability model.

Keywords: Belief functions, quantified beliefs, subjective probabilities, axioms for belief, transferable belief model.

1. Introduction.

1.1. Why representing beliefs is useful?

Why should statisticians, engineers, logicians, philosophers... be interested in a model to represent quantified beliefs? For the statisticians, because statistical inference is essentially building 'beliefs', as done explicitely in Bayesian inference but also in classical inference, significance levels and confidence intervals have a strong flavor of 'belief', even though that 'belief' is supposed to be 'objective'. For the engineer, because building a 'thinking robot' is part of the dream of Artificial Intelligence. To be 'viable' such a robot must be able to reason and to act within an environment ridddled with uncertainty. For the logicians, because approximate and commonsense reasoning are based on beliefs. For the philiosophers, because the representation of belief is part of any search for understanding humans.

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What we philosophically mean by 'belief' is left aside. We study epistemic states involving shades of belief, where the 'belief' express the extend to which an ultimately true or false proposition is believed (Dubois and Prade, 1996). In practice we study the same concept as the one considered by the Bayesian probabilists. The .7 value in the statement 'the probability of A is .7' quantifies someone's 'beliefs'. We use 'belief' in this non categorical sense. We could as well have used words like credibility, support, strength of opinion, necessary commitment... or many other similar expressions. The word 'belief' is just used for simplicity's sake.

Classicaly qantified beliefs is represented by probability functions. This paper reconsiders the justifications that underlie the determination of a model to represent quantified beliefs. Rationality requirements are proposed that lead to the transferable belief model (TBM), a model based on belief functions. The meaning and advantages of the TBM are presented in Smets and Kennes (1994). Such a model was criticized as 'lacking of any axiomatic justification'. We present here such a justification.

The Bayesian model is perfect when all needed probabilities are available. But what when some of these probabilities do not exist? One can either claim that they *always* exist, in which case the Bayesian model covers all needs. This claim hardly makes unanimity. The TBM extends the Bayesian program to those cases where a strict Bayesian approach is questionable if not purely inappropriate. The TBM provides a model much more flexible than the Bayesian model, and reduces itself into the Bayesian model when the conditions underlying the applicability of the Bayesian model are satisfied.

1.2. About the concept of belief.

1) Credal versus pignistic levels. Uncertainty induces beliefs, i.e. graded dispositions that guide our behavior. They manifest themselves at two mental levels: the credal level where beliefs are entertained and the pignistic level where beliefs are used to make decisions[2].

Usually these two levels are not distinguished and probability functions are used to quantify beliefs at both levels. The justification for the use of probability functions is usually linked to "rational" behavior to be held by an ideal agent involved in some betting or decision contexts (Ramsey, 1931, Savage, 1954, DeGroot, 1970). They have shown that if decisions must be "coherent", the uncertainty over the possible outcomes must be represented by a probability function. This result is accepted here, except that such *probability functions quantify the uncertainty only when a decision is really involved*. Therefore uncertainty must be represented by a probability function at the pignistic level. We also accept that this probability function is induced from the beliefs entertained at the credal level. What we reject is the assumption that

[2] Credal and pignistic derive both from the latin words 'credo', I believe and 'pignus', a wage, a bet (Smith, 1961).

this probability function represents the uncertainty at the credal level. We assume that the pignistic and the credal levels are distinct what implies that the justification for using probability functions at the credal level does not hold anymore (Dubois et al., 1996). This paper is concerned with developing the nature of the function that might appropriately represent quantified beliefs at the credal level.

Many comments and disgressions are necessary in order to make the paper self contained but they can be skipped on first reading. These sections are marked **.

**** 2) Historical comments:** The distinction between the two levels was already recognized by previous authors as illustrated by the following quotations. ‘The kind of measurement of belief with which probability is concerned is a measurement of belief *qua* basis of action’ (Ramsey, 1931). ‘That beliefs are necessary ingredients for our decisions does not mean that beliefs cannot be entertained without any revealing behavior manifestations’ (Smith and Jones, 1986, p.147). ‘A belief is a disposition to feel that things are thus-and-so. It must be contrasted with the concept of acceptance’ (Cohen, 1993). ‘A probability measure is a tool for action, not for assessing strength of evidence’ (Sahlin, 1993). ‘There is difference between theoretical reaoning, which immediately modifies beliefs, and practical reasoning, which immediately modifies plans and intentions’ (Harman, 1986).

The normative models proposed to represent quantified beliefs entertained at the credal level can be categorized into three classes:

1) the probabilistic models,

2) the non-standard probabilistic models, i.e., extensions of the probability model: the upper and lower probabilities models (Good, 1950, Smith, 1961, Kyburg, 1987b, Walley, 1991, Voorbraak, 1993), Dempster-Shafer's models (Dempster, 1967, Shafer, 1976, Smets, 1994), the Hints models (Kohlas and Monney, 1994), the probability of provability models (Ruspini, 1986, Pearl, 1988, Smets, 1991), the convex sets of probability functions (Levi, 1980),

3) the non-probabilistic models, i.e., models independent of any underlying probability model: the transferable belief model (Smets, 1988, 1990a, Smets and Kennes, 1994), the possibility theory model (Zadeh, 1978, Dubois and Prade, 1985), etc…

For each of these normative models, authors have proposed sets of requirements that measures of belief should satisfy. The comparison of the requirements helps to evaluate the appropriateness of the normative models.

3) Our strategy for building the model. In this paper, we follow the same strategy. We propose a set of requirements that should be satisfied by the mathematical functions that represent quantified beliefs at the credal level. Therse functions are temporarily called ‘credibility functions’. Initially credibility functions could be any set-functions. We then introduce rationality requirements that any credibility function should satisfy in order to adequately represent beliefs. Each requirement puts more and more constraints on the family of functions that could be used to quantify beliefs, up to the point where that family reduces itself to the set of belief functions. Even though probability functions are special cases of belief functions, the family of probability functions is not rich enough to satisfy all the requirements we introduce.

The resulting model for representing quantified beliefs at the credal level is the TBM (Smets and Kennes, 1994). Like Dempster-Shafer models, the TBM is based on belief functions, but it is free of any probabilistic connotation. As far as we know, this is the first axiomatization based on rationality requirements that justify the use of belief functions to represent quantified beliefs.

4) The evidential corpus. Our approach is normative, not descriptive. We consider beliefs held by an ideal rational agent, denoted You. Your beliefs are relative to the truth status of some propositions. We limit ourselves to propositional logic. Extensions to higher order logics, to multivalued logics and to fuzzy logics are left aside. The strength of the belief entertained by You at time t that a given proposition is true is defined relative to a given evidential corpus, denoted EC_t^Y, i.e., the set of evidence in Your mind at time t. The evidential corpus EC_t^Y corresponds to Your background knowledge, to ‘all what You know at t’. EC_t^Y is the set of information used by You at t to build Your beliefs. It will be constant when we will discuss uninformative refinements and coarsenings. It will change when we will discuss conditioning and deconditioning, i.e., when new pieces of evidence are added (conditioning) or retracted (deconditioning) from EC_t^Y.

**** 5) The strucutre of the evidential corpus.** For what concerns the construction of Your beliefs, EC_t^Y is composed of propositions accepted to be true tby You at t. Some propositions bear directly on the domain on which Your beliefs are built. Other propositions translate rationality principle that Your beliefs should satisfy. For example, as a Bayesian, You would put in EC_t^Y the list of possible events on which Your probabilities will be defined, Your opinion about whch event will prevail, and some rationality requirements like the one that states that the probability given to two mutually exclusive events should be the sum of the probabilities given to the individual events, etc... Given EC_t^Y, if You are a Bayesian, You assign a probability to every event.

One could claim that EC_t^Y uniquely determines these probabilities, a reminiscence of Carnap logical probabilities. Nevertheless, the derived probabilities are subjective as EC_t^Y contains Your personal opinions at t. It is hard to differentiate between subjective probabilities and logical probabilities induced by propositions that describe personal opinions.

The belief set of Gärdenfors (1988) is a subset of EC_t^Y, it is the set of propositions in EC_t^Y that induce the list of possible events on which quantified beliefs will be distributed.

We do not include the beliefs assigned to the various events in EC_t^Y as we want to use it just as a description of the background from which You build Your beliefs at t. If we had put the values of the beliefs in EC_t^Y, then the beliefs induced by EC_t^Y would be nothing but those included in EC_t^Y. We use EC_t^Y as a 'background' and we say 'EC_t^Y induces beliefs so and so' just as we would say 'under such a background knowledge, beliefs are so and so'.

6) Achievement. This paper presents the rationality requirements that should be included in EC_t^Y. Once the rationality requirements are included in EC_t^Y, qantified beliefs are represented by belief functions. The strategy we follow consists in assuming that You is in a given 'belief state' induced by some evidential corpus EC_t^Y. Then we introduce some additional information, like a conditioning information, in Your evidential corpus. Your beliefs must be adapted accordingly in order to keep some 'coherence' between the belief states. We assume Your beliefs are summarized by a 'credibility function'. Therefore the changes between belief states that result from additional information are reflected by transformation between 'credibility functions'. The coherence required among belief states will induce some constraints on the possible nature of the 'credibility functions'. Together the constraints we will introduce imply that 'credibility functions' are belief functions, i.e., that the model for representing quantified beliefs is the TBM.

1.3. Summary of content.

The outline of the paper is as follows. Section 2 sets the frame on which beliefs are held. The frame is essentially a finite Boolean algebra over a set of possible worlds. Credibility functions are required to be bounded real valued functions, monotone for inclusion, that give the same belief to two propositions that are considered as equivalent by You.

Section 3 shows that the set of credibility functions is convex. Section 4 introduces the concepts of uninformative coarsening and refinements, i.e., changes limited to the granularity of the domain on which credibility functions are defined. Section 5, by far the most importanty section, studies the impact of a new piece of evidence on a credibility function, i.e., the conditioning process. Section 6 introduces the concept of deconditionalization and shows that a credibility function is in fact a belief function. It concludes our justification for the use of belief functions to quantify beliefs. Section 7 summarizes the results and answers potential questions. Proofs are given in the appendix.

2. The credibility function.

The aim of this work is to develop the mathematical structure of a function Cr, called a credibility function, that quantifies Your beliefs. The kind of belief we are concerned with are those encountered in statements like 'Your belief that a proposition A is true is .7', or as a shortcut, 'Your belief that A is .7'. Beliefs weight the strength given by the agent You to the fact that a given proposition is true in the actual world. The meaning of the statement 'Your belief that it will rain tomorrow is .7' can be either: 'the measure of the belief held by You that the proposition "it will rain tomorrow" is true is .7' or: 'You believe at level .7 that the day of tomorrow belongs to the set of rainy days'. So beliefs given to 'propositions' can equivalently be given to the subsets of worlds that denote the propositions. Defining beliefs on propositions or on sets is equivalent. We will adopt the second approach. We restrict ourselves to propositional logic, therefore the sets of worlds are also equivalent to the 'events' considered in probability theory.

We proceed now by formally defining the domain on which You will express Your beliefs at time t given Your evidential corpus EC_t^Y. Then a first set of requirements are presented, the most important being the doxastic consistency, i.e., propositions that are considered as equivalent by You at t given EC_t^Y should receive equal beliefs.

2.1. The propositional space.

We formalize the domain on which degrees of belief will be assessed. The domain, called the credibility domain, will be a Boolean algebra built on a set of possible worlds.

1) Possible worlds. Our presentation is based on possible worlds (Carnap, 1962, Ruspini, 1986, Bradley and Swartz, 1979) and beliefs will be given to sets of worlds. These sets of worlds, called 'events' in probability theory, will be elements of a Boolean algebra of sets.

Let L be a finite propositional language, supplemented by the tautology and the contradiction, denoted $\top$ and $\bot$, respectively. Let Ω_L be the set of worlds that correspond to the interpretations of L and built so that every world corresponds to a different interpretation. Propositions identify the subsets of Ω_L, and the subsets of Ω_L denote propositions. For any proposition X, let $[\![X]\!]\subseteq\Omega_L$ be the set of worlds identified by X (i.e., those worlds where X is true).

We assume that among the worlds of Ω_L a particular one, denoted ω_0, corresponds to the actual world. You ignore at t which world is ω_0. You can only express Your beliefs at t that the actual world ω_0 belongs or not to this or that subsets of Ω_L.

2) The frame of discernment. By definition the actual world ω_0 is an element of Ω_L. But because of Your limited understanding of Ω_L, some of the worlds of Ω_L might be not conceivable to You at t. Let $\Omega\subseteq\Omega_L$ be the set of worlds conceived by You at t given Your evidential corpus EC_t^Y. The set Ω is called the frame of discernment.

**** 3) The structure of the frame of discernment.** The set Ω results from EC_t^Y and contains all the worlds conceivable by You at t given EC_t^Y. Of course EC_t^Y can say more about Ω. It can tell that some worlds in Ω are in fact considered as impossible by You at t. Let $[\![EC_t^Y]\!]$ denote the set of worlds in Ω where all the propositions deduced on L from EC_t^Y are true. Hence Your beliefs are essentially defined on $[\![EC_t^Y]\!]$ as, at time t, You consider those worlds in Ω and not in $[\![EC_t^Y]\!]$ as impossible. Nevertheless we can innocuously extend the domain of Your beliefs to Ω. So by construction, $[\![EC_t^Y]\!]\subseteq\Omega$. The worlds in Ω and not in

$[\![EC_t^Y]\!]$ are considered as impossible to You at t, the worlds in Ω_L and not in Ω are inconceivable to You at t: impossible and inconceivable worlds should not be confused.

Of course, defining Ω as $[\![EC_t^Y]\!]$ could also be accepted as beliefs are always allocated to subsets of $[\![EC_t^Y]\!]$. So distinguishing between Ω and $[\![EC_t^Y]\!]$ is not really important.

Classically, inconceivable worlds are not considered and $\Omega = \Omega_L$. A difference between Ω and Ω_L can nevertheless appear if You had built Ω by an enumeration procedure, and You had omitted (because of Your limited understanding) to list some of the possible worlds.

Your beliefs about ω_0 can only be expressed for the subsets of Ω. The idea of speaking about the belief given by You to a set of worlds inconceivable to You seems difficult to accept and is thus rejected. Note that nothing requires ω_0 to be in Ω: the actual world can be one of those worlds 'inconceivable' to You at t.

When $\Omega \neq \Omega_L$, it could be tempting to consider the set η of worlds of Ω_L not in Ω, and to define Your beliefs on Ω_L. We prefer to avoid such artifice as we feel that You could be in a state of beliefs where he can only express Your beliefs over the subsets of Ω. Creating the extra set η works innocuously in probability theory, but not with more general theories where the degree of belief given to $A \cup \eta$ for $A \subseteq \Omega$ is not just the sum of the degrees of belief given to A and to η. In these more general theories, if we add the extra set η, You would have to specifically assess Your beliefs for the subsets $A \cup \eta$ for all $A \subseteq \Omega$. This is not realistic as You do not know what these subsets represent as those worlds in η are 'inconceivable' for You at t.

It is worth noticing that extending the belief domain from $[\![EC_t^Y]\!]$ to Ω was accepted, whereas extending it from Ω to Ω_L was not. The reason for such asymmetry is that, in the first case, You know that the worlds in Ω not in $[\![EC_t^Y]\!]$ are impossible, whereas You have no opinion about the worlds in Ω_L not in Ω.

4) Doxastic equivalence. In the propositional language L, two propositions are logically equivalent iff the sets of worlds that denote them are equal. Besides this logical equivalence, there is another form of equivalence that concerns Your beliefs. Suppose You want to decide whether to go to a movie or stay at home tonight. You have decided to toss a coin, and if it is heads, You will go to the movie, and if it is tails, You will stay at home. (These are the pieces of evidence in EC_t^Y). Then 'heads' and 'going to the movie' are 'equivalent' from Your point of view as they share the same truth status given what You know at t. Of course, they are not logically equivalent (Kyburg, 1987a). We call them doxastically equivalent (from doxa = an opinion, in Greek). Logical equivalence implies doxastic equivalence, not the reverse.

Definition: Two propositions p and q defined on L are doxastically equivalent (for You at t, i.e., given EC_t^Y) iff the sets of worlds $[\![p]\!]$ and $[\![q]\!]$, both subsets of Ω_L, that denote them share the same worlds among those in $[\![EC_t^Y]\!]$, i.e., $[\![EC_t^Y]\!] \cap [\![p]\!] = [\![EC_t^Y]\!] \cap [\![q]\!]$.

Doxastic equivalence of propositions p and q under EC_t^Y is denoted by: $[\![p]\!] =_{EC_t^Y} [\![q]\!]$.

5) Complement. For $A \subseteq \Omega$, $\overline{A}$ denotes the set of worlds in Ω not in A. By definition, $A \cup \overline{A} = \Omega$.

6) The propositional space. Whenever You can express Your belief that ω_0 belongs to a set A, and to a set B, You can also express Your belief that ω_0 belongs to their complement (relative to Ω), union and intersection. Therefore, the domain of Your beliefs is assumed to be a Boolean algebra of subsets of the frame of discernment Ω (thus closed under union, intersection, complement, and containing Ω and $\emptyset$).

Let $\Re$ denote this Boolean algebra of subsets of Ω on which You can express Your beliefs. We call the pair $(\Omega, \Re)$ a propositional space and $\Re$ the *credibility domain*. The *atoms* of the credibility domain $\Re$ are defined as the 'smallest' non empty elements of $\Re$ such that their intersection with any element of $\Re$ is either themselves or the empty set. Let $At(\Re)$ denoted the set of atoms of $\Re$. Note that several worlds of Ω might belong to one atom of $\Re$. The atoms of $\Re$ are in fact the elements of a partition of Ω. When $\Re$ is the power set of Ω, the atoms of $\Re$ are the singletons of Ω. Given $\Re$, the number of atoms in a set $A \in \Re$, denoted $|A|$, is the number of atoms on $\Re$ that are included in A.

**** 7) Details about the propositional space.** Why do we introduce the credibility domain $\Re$, restricting Your beliefs to it, and we just do not accept that $\Re$ is the power set of Ω? The reason is that the propositional language L can be very rich, therefore the worlds of Ω can denote very precise propositions, and due to Your limited understanding, You cannot express Your beliefs on such a detailed domain. When You wants to assess Your beliefs about tomorrow weather in Brussels, he will not asses Your beliefs on the weather at every geographical location. He will restrict himself to Brussels even though L could be: {'Brussels weather is fine', 'New York weather is fine', 'Tokyo weather is fine', ...}. When asked about Your belief about Brussels weather, You builds a credibility domain $\Re$ with two atoms: one where Brussels weather is fine, and one where Brussels weather is not fine. He will not build an atom where simultaneously Brussels weather is fine and New York weather is not fine and Tokyo weather is fine, etc... You just does not care about such a refined domain.

As an example of a propositional space $(\Omega, \Re)$ and of the atoms of $\Re$, consider a given person X whose gender and pregnancy status You wonders about. Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ where X is a pregnant female in ω_1, a non-pregnant female in ω_2, and a non-pregnant male in ω_3. There is no ω_4 world where X is a pregnant male because ω_4 is not conceivable for You at t. Let $\Re_1$ and $\Re_2$ be two credibility domains on Ω with $At(\Re_1)$ = {Female, Male} = $\{\{\omega_1,$

ω_2}, {ω_3}} and At($\Re_2$) = {Pregnant, Non-pregnant} = {{ω_1}, {ω_2, ω_3}}. Then $\Re_1$ is {{}, Female, Male, Female$\cup$Male} = {{}, {ω_1, ω_2}, {ω_3}, {ω_1, ω_2, ω_3}}and $\Re_2$ = {{}, Pregnant, Non-pregnant, Pregnant$\cup$Non-pregnant} = {{}, {ω_1}, {ω_2, ω_3}, {ω_1, ω_2, ω_3}}. In $\Re_1$, $|\Omega|$=2. If $\Re$ is the power set of Ω, then $|\Omega|$ = 3.

8) Unique agent. Only one agent You is considered in this paper, and time t is unique except when belief revision is studied. It should nevertheless be remembered that the evidential corpus EC_t^Y, the frame of discernment Ω, the credibility domain $\Re$, and the credibility function Cr to be soon introduced are all relative to You at t.

2.2. The credibility function.

Let (Ω, $\Re$) be a propositional space and You be an agent with EC_t^Y being Your evidential corpus at time t. We first require that the beliefs held by You at time t are quantified by a point-valued 'credibility' function Cr which maps $\Re$ into a closed interval of the real line, is monotone for inclusion, and reaches its lower limit for Ø.

Requirement A2.1: Let (Ω, $\Re$) be a propositional space. Let EC_t^Y be agent Your evidential corpus at time t. Then Your beliefs allocated to the elements of $\Re$ given EC_t^Y are quantified by a function Cr, where

1: $Cr : \Re \rightarrow [\alpha_\perp, \alpha_T]$ where $[\alpha_\perp, \alpha_T]$ is an interval of the real line.

2: $\forall A, B \in \Re$, if $A \subseteq B$, then $Cr(A) \leq Cr(B)$

3: $Cr(\emptyset) = \alpha_\perp$.

The triple (Ω, $\Re$, Cr) is called a *credibility space*. Requirement A2.1 is already very strong as it eliminates models based on sets of probability functions (Kyburg, 1987b, 1995, Voorbraak, 1993, Levi, 1980) or on interval valued probabilities (Walley, 1991).

The *belief state* of You at t about the frame of discernment Ω is defined by the quadruple (Ω, $\Re$, Cr, EC_t^Y). The credibility domain $\Re$ is the algebra that bears Your beliefs and Cr is the function that assigns to every element A of $\Re$ a value that quantifies Your belief that the actual world ω_0 belongs to A.

Given Ω and EC_t^Y, You can build several credibility domains $\Re_i$, i=1,2..., and build a credibility function Cr_i on each of these credibility domains. Hence given Ω and EC_t^Y, there exists a family of credibility spaces (Ω, $\Re_i$, Cr_i) that will represent Your beliefs. The next requirement is proposed in order to preserve coherence between these credibility spaces. It states that given EC_t^Y, two doxastically equivalent propositions should receive the same credibility (Kyburg, 1987a).

Requirement A2.2: Doxastic Consistency.

Let $(\Omega, \Re_i, Cr_i, EC_t^Y)$, i=1,2, be two belief states based on the same EC_t^Y and relative to two credibility domains built on the same frame of discernment Ω. Let $A_1 \in \Re_1$, $A_2 \in \Re_2$.

If $A_1 =_{EC_t^Y} A_2$, then $Cr_1(A_1) = Cr_2(A_2)$.

Requirement A2.2 implies that those subsets of Ω that belong to both $\Re_1$ and $\Re_2$ will receive the same belief. Indeed the propositions that identify these subsets are doxastically equivalent for You at t. A consequence of A2.2 is that the belief given by You at t to a subset of Ω does not depend on the structure of the algebra to which the subset belongs.

3. Convexity of the set of credibility functions.

In this section, we show that the set of credibility functions defined on a credibility domain is a convex set. Such a property is needed as some requirements use the property that the convex combination of two credibility functions is a credibility function. Along the way, we assume that probability functions are credibility functions, thus making the classical probability theory at least a special case of our general model.

** 3.1. Example.

The next example illustrates the origin of the convexity property.

Example 1: Horse Race. Suppose a horse race involving three horses: Allan, Blues and Carol. Tomorrow at 7 AM, it will be decided depending on the outcome of a coin tossing experiment, if the race will be run at 9 AM or 11 AM. Let α be the probability that the race is run at 9 AM. The time of the race influences Your beliefs about which horse will win. Let Cr_1 and Cr_2 be the credibility functions that describe Your beliefs about which horse will win if the race is run at 9 AM or at 11 AM, respectively. You must buy a ticket now. Let Cr_{12} be the credibility function that describes Your beliefs held by now about the winner not knowing at which time the race will be run. We essentially assume that $Cr_{12}(A)$ for A⊆{Allan, Blues, Carol} depends only on $Cr_1(A)$, $Cr_2(A)$ and α. □

So we have two belief states $(\Omega, \Re, Cr_1, EC_t^Y)$ and $(\Omega, \Re, Cr_2, EC_t^Y)$ where Ω = {Allan, Blues, Carol}, and ℜ is the power sets of Ω. One of the two belief states will be selected by a chance process. Let $(\Omega, \Re, Cr_{12}, EC_t^Y)$ be the belief state built on ℜ before learning which belief state will prevail. What are the coherence requirements to impose on Cr_{12}, i.e., what is the relation between Cr_{12} and Cr_1 and Cr_2. We want:

1) Cr_{12}({Allan}) depends only on Cr_1({Allan}) and Cr_2({Allan}) (and α), Cr_{12}({Allan, Blues}) depends only on Cr_1({Allan, Blues}) and Cr_2({Allan, Blues}) (and α), etc...,

2) Cr_{12} strictly and continuously increases when Cr_1 and/or Cr_2 increase,

3) if Cr_1({Allan}) = Cr_2({Allan}), then Cr_{12}({Allan}) should also be equal to Cr_1({Allan}), etc...

These requirements are sufficient to derive very strong results on the relation between Cr_{12} and its components Cr_1 and Cr_2. □

**** 3.2. Convexity Requirements.**

The following requirements are postulated for the credibility function Cr_{12}.

Requirement A3.1: Pointwise Compositionality.
There exists a function $F_\alpha:[\alpha_\perp, \alpha_T]^2 \to [\alpha_\perp, \alpha_T]$ such that for each $A \in \Re$,

$$Cr_{12}(A) = F_\alpha(Cr_1(A) , Cr_2(A)).$$

Requirement A3.2: Continuity.

$F_\alpha(x,y)$ is continuous in $(x,y) \in [\alpha_\perp, \alpha_T]^2$.

Requirement A3.3: Strict Monotonicity.

$F_\alpha(x,y)$ is strictly monotone for x and $y \in [\alpha_\perp, \alpha_T]$.

Requirement A3.4: Idempotency.

$F_\alpha(x,x) = x$ for all $x \in [\alpha_\perp, \alpha_T]$.

The Pointwise Compositionality, reflected by the existence of the F function, is justified by the idea that, for a given $A \in \Re$, $Cr_{12}(A)$ should not be changed if we replace the credibility domain $\Re$ by the credibility domain $\Re'$ with only two atoms, A and $\bar{A}$. Requiring that Cr_{12} should be unchanged after regrouping the atoms is obvious (and will be fully analyzed in section 4.1). The real limitation introduced by requirement A3.1 is that we do not have:

$$Cr_{12}(A) = F_\alpha(Cr_1(A) , Cr_2(A) , Cr_1(\bar{A}) , Cr_2(\bar{A})),$$

but we consider only the first two terms. It translated the idea that $Cr_{12}(A)$ should only depend on the beliefs given to those propositions that enter in the construction of A.

The other requirements are hardly arguable. Continuity is assumed essentially for simplicity sake, and strict monotonicity is postulated as we consider that Cr_{12} should be sensitive to both its arguments. Idempotency reflects the idea that if the beliefs do not depend on the time when the race is run, i.e., $Cr_1 = Cr_2$, then Cr_{12} should be equal to Cr_1.

We now show that F satisfies the bisymmetry equation:

$$F_\alpha(F_\alpha(x,y) , F_\alpha(u,v)) = F_\alpha(F_\alpha(x,u) , F_\alpha(y,v)), \qquad (3.1)$$

which solution is analyzed in Aczel (1966, pg. 287). The origin of the bisymmetry equation is illustrated in example 1, continuation 1.

Example 1, continuation 1. Suppose that the race could be run at 9 AM, 11 AM, 3 PM or 5 PM. The choice of the time will be made by applying a random device to decide between AM or PM. Let α be the probability that the race is run in the morning. Once selection between AM and PM have been achieved, the race organizers use the same random device to select between the first or second time slot, with α being the probability that the race is run at 9 AM if run in the morning, or 3 PM if run in the afternoon. Independence is assumed. So the probability that the race is run at 9 AM is α^2, at 11 AM, $\alpha(1-\alpha)$, at 3 PM, $(1-\alpha)\alpha$, at 5 PM, $(1-\alpha)^2$.

Let $(\Omega, \mathfrak{R}, Cr_i, EC_t^Y)$, i=1,2,3,4, be the belief states that represent Your belief about the winner depending if the race is run at 9 AM, 11 AM, 3 PM or 5 PM, respectively. We want to build $(\Omega, \mathfrak{R}, Cr_{1234}, EC_t^Y)$, Your belief state before deciding when the race will be run. There are two identical ways to build Cr_{1234}. You can first build Cr_{12} from Cr_1 and Cr_2 where the probability that Cr_1 will prevail is α, and Cr_{34} from Cr_3 and Cr_4 where the probability that Cr_3 will prevail is α, and then build Cr_{1234} from Cr_{12} and Cr_{34} where the probability that Cr_{12} will prevail is α. You can as well build Cr_{13} from Cr_1 and Cr_3 where the probability that Cr_1 will prevail is α, and Cr_{24} from Cr_2 and Cr_4 where the probability that Cr_2 will prevail is α, and then build Cr_{1234} from $Cr1_{13}$ and Cr_{24} where the probability that Cr_{13} will prevail is α. The choice of the selection procedure was specifically done so that all the mentioned probabilities are equal. Both approaches should lead to the same final results. This requirement translates into: for each $A \in \mathfrak{R}$,

$$Cr_{12}(A) = F_\alpha(Cr_1(A) , Cr_2(A))$$
$$Cr_{34}(A) = F_\alpha(Cr_3(A) , Cr_4(A))$$
$$Cr_{1234}(A) = F_\alpha(Cr_{12}(A) , Cr_{34}(A))$$

$$Cr_{13}(A) = F_\alpha(Cr_1(A) , Cr_3(A))$$
$$Cr_{24}(A) = F_\alpha(Cr_2(A) , Cr_4(A))$$
$$Cr_{1234}(A) = F_\alpha(Cr_{13}(A) , Cr_{24}(A)).$$

From the equality between Cr_{1234}, we have:

$$Cr_{1234}(A) = F_\alpha(F_\alpha(Cr_1(A) , Cr_2(A)) , F_\alpha(Cr_3(A) , Cr_4(A)))$$
$$= F_\alpha(F_\alpha(Cr_1(A) , Cr_3(A)) , F_\alpha(Cr_2(A) , Cr_4(A))).$$

This is the bisymmetry equation (3.1). □

Theorem 1: Given requirements A3.1 to A3.4, the F_α function in requirement A3.1 that satisfies the bisymmetry equation (3.1) is of the form:

$$F_\alpha(x,y) = f_\alpha(a.f_\alpha^{-1}(x) + (1-a).f_\alpha^{-1}(y)) \quad (3.2)$$

with continuous, strictly monotone $f_\alpha: [\alpha_\perp,\alpha_T]\rightarrow[\alpha_\perp,\alpha_T]$, $f_\alpha^{-1}(0) = \alpha_\perp$, $f_\alpha^{-1}(1) = \alpha_T$ and $a\in[0, 1]$ where a may depend on α.

Proof: see Aczel (1966, page 287).

**** 3.3. Canonical scale.**

We assume that probability functions are credibility functions. Such an assumption acknowledges that Bayesian theory should be at least a subset of the theory we are developing.

Requirement A3.5: Probability functions belong to the set of credibility functions.

Assuming probability functions are credibility functions, one obtains $\alpha_{\perp} = 0$ and $\alpha_{T} = 1$, as impossible event has a probability 0, sure event has probability 1, and these are the extreme values belief can take. So $f_{\alpha}^{-1}(0) = 0$ and $f_{\alpha}^{-1}(1) = 1$.

When building Cr_{12} from Cr_1 and Cr_2 , suppose the wo credibility functions Cr_1 and Cr_2 are probability functions P_1 and P_2. These two probability functions are conditional probability functions. Then by probability calculus we know that Cr_{12} is also a probability function, and:

$$Cr_{12} (A) = \alpha\, P_1(A) + (1-\alpha)\, P_2(A).$$

In particular with $P_1(A) = 1$, $P_2(A) = P_3(A) = P_4(A) = 0$, we obtain from (3.2):

$$\alpha = f_{\alpha}(a)$$

and $\alpha^2 = f_{\alpha}(a^2)$

Iterating wuth 6, 8 ...2n possible times for the race in example 1, we obtain:

$$\alpha^n = f_{\alpha}(a^n),$$

hence: $f_{\alpha}(a^n) = (f_{\alpha}(a))^n$,

an equation which unique solution is: $f_{\alpha}(a) = a$.

In that case $a = \alpha$.

Relation (3.2) becomes: $F_{\alpha}(x,y) = \alpha.x + (1-\alpha).y$

Theorem 2 summarizes the results of section 3.

Theorem 2: Given the requirements A2 and A3, the set of credibility functions defined on the same credibility domain is a convex set, and $\alpha_{\perp} = 0$ and $\alpha_{T} = 1$

4. Coarsening and Refinement.

4.1. Example.

We study the impact on Your beliefs that would result from an ‘uninformative’ change of the credibility domain on which Cr is initially defined. We consider two types of changes: the coarsening and the refinement. Intuitively the first corresponds to a grouping together of the atoms of $\Re$ whereas the second corresponds to a splitting of the atoms of $\Re$ (see figure 1). The next example motivates the concept of uninformative changes of the credibility domain.

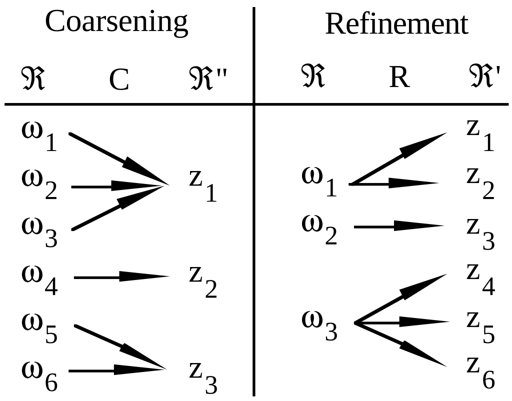

Figure 1: Examples of a coarsening C from ℜ to ℜ” and a refinement R from ℜ to ℜ’ whose atoms are the ω_i's and the z_i's, respectively.

Example 2: Killer's Nationality. Suppose a person has been murdered. Let Cr_0 represent Your beliefs that the killer (k) is English, German, French or Italian. Cr_0 is defined on the subsets of {E, G, F, I}. We consider how Cr_0 will be adapted when the domain of Your belief is changed. Two transformations are considered: coarsening and refinement. In the first case, suppose French and Italian are grouped into the set ‘Mediterranean’. The new space {E, G, M} is a coarsening of the initial space. In the second case, suppose the set ‘French’ is partitioned into two subsets, the sets ‘FrenchTuc’ (FT) and ‘FrenchPic’ (FP). The new space {E, G, FT, FP, I} is a refinement of the initial space. These transformations of the frames on which Your beliefs are defined are said to be 'uninformative' inasmuch as Your evidential corpus EC_t^Y is unchanged for what concerns Your beliefs about the killer's nationality. To change the granularity of the frames does not modify Your beliefs for those subsets that are doxastically equivalent.

Let Cr_1 and Cr_2 represent Your belief on {E, G, M} and {E, G, FT, FP, I}, respectively. By Doxastic Consistency, $Cr_1(E) = Cr_0(E)$, $Cr_1(M) = Cr_0(F\cup I)$, etc... and in fact Cr_1 is entirely defined from Cr_0. Identically, $Cr_2(E) = Cr_0(E)$, $Cr_2(FT\cup FP) = Cr_0(F)$, ... but not all values of Cr_2 are derivable by Doxastic Consistency from Cr_0: so it is the case for $Cr_2(FT)$, $Cr_2(F,\cup E)$... Hence extra requirements will be introduced. □

** 4.2. Formal definitions.

Formally, we have the next definitions.

Coarsening: Let (Ω, ℜ) be a propositional space. A coarsening C is a mapping from ℜ to ℜ”, where ℜ” is an algebra also defined on Ω, such that one or several atoms of ℜ are mapped into one atom of ℜ” and each atom of ℜ is mapped into one and only one atom of ℜ”.

Let C(ω) be the atom of ℜ” on which the atom ω of ℜ is mapped, and $\forall A\in\Re$, $C(A) = \bigcup_{\omega\in At(A)} C(\omega)$ where At(A) are the atoms of ℜ subsets of A. For A∈ℜ”, $C^{-1}(A)$ is the union of the atoms of ℜ which are mapped by C into an atom of A.

Refinement: Let (Ω, ℜ) be a propositional space. A refinement R is a mapping from ℜ to ℜ' where ℜ' is an algebra on Ω' such that each atom of ℜ is mapped into one or several atoms of ℜ' and each atom of ℜ' is derived from one and only one atom of ℜ. Let R(A) be the image of A∈ℜ in ℜ', and let R(Ø)=Ø.

In a belief state (Ω, ℜ, Cr, EC_t^Y), the structure of the frame of discernment Ω is, in fact, not essential. The only relevant component in the propositional space is the algebra ℜ. Hence in a refinement, we only require that Ω’ is sufficiently detailed so that each atom of the algebra ℜ’ can be uniquely defined. Therefore we will always define Ω and Ω’ such that they are equal and the atoms of the algebras ℜ and ℜ’ can be defined from the elements of Ω.

4.3. The uninformativeness requirement.

Given a belief state (Ω, ℜ, Cr, EC_t^Y), we want to build the belief states (Ω, ℜ’, Cr’, EC_t^Y) and (Ω, ℜ”, Cr”, EC_t^Y) in a coherent way. The coarsenings and refinements are called uninformative because the evidential corpus EC_t^Y held by You at t stays unchanged. Uninformative changes fit in with the idea that only the structure of the algebras on which beliefs are held is modified, no further information is added to the evidential corpus.

The uninformative nature of the changes and the coherence we ask for are formalized in the next requirement that states that the credibility functions encountered in the belief states induced by such mappings are completely determined by Cr and the mappings.

Requirement A4.1: Let (Ω, ℜ, Cr, EC_t^Y) be a belief state. Let R be an uninformative refinement from (Ω, ℜ) to (Ω, ℜ’) and let C be an uninformative coarsening from (Ω, ℜ) to (Ω, ℜ”). Let the belief states (Ω, ℜ’, Cr’, EC_t^Y) and (Ω, ℜ”, Cr”, EC_t^Y). Then Cr’ and Cr” are completely determined by Cr and R and C, respectively. So there are g and h functions such that:

$$Cr' = g(Cr, R) \qquad \text{and} \qquad Cr'' = h(Cr, C).$$

4.4. Uninformative Coarsening.

The derivation of the nature of the h transformation is immediate. In figure 1, $\{z_1\}$ is doxastically equivalent to $\{\omega_1, \omega_2, \omega_3\}$, hence they share the same credibility. Identically, $\{z_1, z_2\}$ is doxastically equivalent to $\{\omega_1, \omega_2, \omega_3, \omega_4\}$, etc… The credibility function over ℜ automatically induces the credibility function on ℜ”, as given in theorem 3.

Theorem 3. Let (Ω, $\Re$", Cr", EC_t^Y) be the belief state derived from the belief state (Ω, $\Re$, Cr, EC_t^Y) by the uninformative coarsening C from (Ω, $\Re$) to (Ω, $\Re$"). Given requirements A2, A3 and A4.1,

$$Cr''(A) = Cr(C^{-1}(A)) \qquad \text{for all } A \in \Re \tag{4.1}$$

Let v be the Möbius transform of Cr (see appendix 1). Then the Möbius transform v" of Cr" is such that,

$$v''(A) = \sum_{B: B \in \Re, C(B)=A} v(B) \qquad \text{for all } A \in \Re'' \tag{4.2}$$

The effect of the coarsening results in an additive transfer of v(B) to the 'smallest' element of $\Re$" that contains B (where smallest means ‘with the smallest number of atoms’). The only difference between Cr" and Cr resides in the fact that Cr provides a more detailed information on Ω than Cr". Indeed Cr describes a belief over an algebra $\Re$ whose granularity is finer than the one of $\Re$".

4.5. Uninformative refinement.

We illustrate the uninformative refinement in the next example.

Example 3. Failure diagnosis.
Suppose an electrical equipment has failed and You knows that one and only one circuit has failed. There are two types of circuits, the A- and the B-circuits made at the F_A and F_B factories, respectively. You knows that circuits made at factory F_A are of high quality whereas those at factory F_B are of a lower quality. Hence You might have good reasons to believe that the broken circuit is a B-circuit, even though it might be a A-circuit. The belief state of You is denoted by (Ω, $\Re$, Cr_0, EC_0)[3], where $\Re$ is the power set of {A, B}[4], and Cr_0 represents Your belief about which type of circuit is broken, with $Cr_0(A)$ and $Cr_0(B)$ being the degree of belief given by You to the fact that the broken circuit is an A- or a B-circuit, respectively. The atoms of the algebra $\Re$ on which Your beliefs are assessed are:

$$At(\Re) = \{\{A\}, \{B\}\}.$$

Then You learns that the A-circuits are painted in green (G) and the B-circuits are painted in white (W) and pink (P). Let $\Re$' be the power set of {G, W, P}. By construction, $\Re$' results from a refinement R of $\Re$, with R(A) = G and R(B) = P$\cup$W. For You, the color has nothing to do with failure (as far as You knows), thus from Your point of view, R is an uninformative refinement. Let (Ω, $\Re$', Cr', EC_0) be the belief state of You where Cr' quantifies Your beliefs about the color of the broken circuit. The uninformative nature of the refinement is reflected by the fact that the evidential corpus EC_0 has not changed. This results from the assumption that

[3] For simplicity’s sake, we drop the explicit index Y in the evidential corpora, even though all evidential corpora should be understood as depending on Y.
[4] As already mentioned, the detailed nature of Ω is irrelevant. Ω must only be so defined that the atoms of the algebras built on it can be defined from the elements of Ω.

the information relative to the color does not change Your knowledge for what concerns which is the broken circuit. By Doxastic Consistency, $Cr'(G) = Cr_0(A)$, $Cr'(P\cup W) = Cr_0(B)$ and $Cr'(G\cup P\cup W) = Cr_0(A\cup B)$, but $Cr'(P)$, $Cr'(W)$, $Cr'(G\cup P)$, $Cr'(G\cup W)$ are still undefined except for the inequalities that result from the monotonicity of Cr'. For instance:

$$0 \leq Cr'(P) \leq Cr'(P\cup W) = Cr_0(B)$$

and $$Cr_0(A) = Cr'(G) \leq Cr'(G\cup P) \leq Cr'(G\cup P\cup W) = Cr_0(A\cup B).$$ □

The combination of iterated refinements and Doxastic Consistency Principle (A2.2) allows us to greatly simplify the probelm of deriving the impact of an uninformative refinement on the credibility functions. Let $(\Omega, \Re_0, Cr_0, EC_t^Y)$ be Your belief state at time t . Suppose two uninformative refinements R_1 and R_2 from $\Re_0$ to $\Re_1$, and $\Re_1$ to $\Re_2$, respectively. Let the refinement R_{12} from $\Re_0$ to $\Re_2$ be defined as the refinement obtained by applying successively R_1, and then R_2: $R_{12} = R_2 \circ R_1$. Let Cr_1, Cr_2 and Cr_{12} be the credibility functions induced on the algebra $\Re_1$, $\Re_2$ and $\Re_2$ by the uninformative refinements R_1, R_2 and R_{12}, respectively. By the Doxastic Consistency Principle, Cr_{12} is equal to Cr_2. Indeed they concern the same propositional space and the evidential corpus EC_t^Y has not changed. The only difference is how Cr_2 and Cr_{12} were built, iteratively for Cr_2 and directly for Cr_{12}. This property highly simplifies proofs.

Let us call 'elementary' the refinement where only one atom of the initial space is split into two atoms in the refined space, all other being kept unsplit. We say that an elementary refinement acts on ω if ω is the atom to be split: we denote it ωR. Every refinement can be represented by some sequence of elementary refinements.

In order to derive the impact of the refinement process on a credibility function, it is sufficeint to study the impact of an elementary refinement. Studying the evolution of the credibility functions through the elementary refinements will allow us to find the overal impact of any refinement.

Another consequence of the Doxastic Consistency requirement is that the credibility function derived from an uninformative refinement is fully characterized by only a few terms of the initial credibility function. To see it, consider the belief state $(\Omega, \Re, Cr, EC_t^Y)$. For $\omega \in \Re$, let ωR be an uninformative elementary refinement from $\Re$ to $\Re'$ with $\omega R(\omega) = \{\omega_1, \omega_2\}$. Let $A \in \Re$ with $A \cap \omega = \emptyset$, and let $B = \overline{\omega \cup A}$. We then build the coarsening C of $\Re$ such that the atoms of the resulting algebra $\Re''$ are {ω, A, B}. Let the elementary refinement ωR" from $\Re''$ to $\Re^*$. The credibility function Cr* derived from Cr by the coarsening C followed by the refinement ωR" depends only on CrC (requirement A4.1). Let Cr_C be the credibility function induced from Cr after applying the coarsening C and let Cr* be the credibility function induced from Cr_C after applying the refinement ωR". Let Cr_R be the credibility function derived from Cr on $\Re'$ by ωR. By Doxastic Consistency, $Cr_R(A\cup\omega) = Cr^*(A\cup\omega)$. As far as $Cr^*(A\cup\omega)$ depends only on Cr_C, so does $Cr_R(A\cup\omega)$. Hence there is a g fiunction so that:

$$Cr_R(A\cup\omega) = g(A\cup\omega, Cr(\omega), Cr(A), Cr(B), Cr(A\cup\omega), Cr(B\cup\omega), Cr(A\cup B), Cr(\Omega))$$

To illustrate the next equality, suppose $\Re_0$ has only two atoms ω_{01} and ω_{02}, and let R_1 and R_2 be elementary refinements (see figure XX) with:
$R_1(\omega_{01}) = \{\omega_{11}, \omega_{12}\}$, $R_1(\omega_{02}) = \{\omega_{02}\}$,
$R_2(\omega_{11}) = \{\omega_{21}, \omega_{22}\}$, $R_2(\omega_{12}) = \{\omega_{12}\}$ and $R_2(\omega_{02}) = \{\omega_{02}\}$.

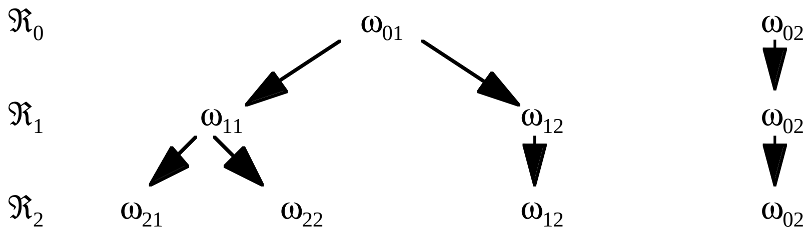

Figure XX. Iterated refinement used for the analysis.

We assume that $Cr_1(\omega_{11}) = Cr_1(\omega_{12})$ and $Cr_1(\omega_{02}\cup\omega_{11}) = Cr_1(\omega_{02}\cup\omega_{12})$. This assumption could be assimilated to a principle of Indifference under Refinement that translates the idea of uninformativeness for a refinement. Its generalization is given in the next requirement.

Requirement A4.2: Indifference under Refinement.
Let $(\Omega, \Re, Cr, EC_t^Y)$ be a belief state. Let R be an uninformative elementary refinement from $(\Omega, \Re)$ to $(\Omega', \Re')$ acting on atom ω of $\Re$. Let $B\in\Re'$ where $B\cap R(\omega) = \emptyset$. Let $R(\omega) = \{\omega_1, \omega_2\}$. Let Cr' be the credibility function derived from Cr on $\Re'$ by R. Then:

$$Cr'(B\cup\omega_1) = Cr'(B\cup\omega_2).$$

A consequence of this principle and the comments about the iterated refinement is that:

$$Cr_{12}(\omega_{21}) = Cr_{12}(\omega_{22}) = Cr_{12}(\omega_{12})$$

and $Cr_1(\omega_{11}) = Cr_1(\omega_{12})$.

We also have: $Cr_2(\omega_{12}) = Cr_1(\omega_{12})$

and $Cr_2(\omega_{11}\cup\omega_{21}) = Cr_1(\omega_{11})$.

Hence: $Cr_2(\omega_{21}) = Cr_2(\omega_{22}) = Cr_2(\omega_{11}\cup\omega_{21})$,

and similar relations that can be written as $Cr(A) = Cr(B) = Cr(A\cup B)$, an equality normally not satisfiable in probability theory. In fact, uninformative refinement is at the core of the divergence between our model and the probabilist models.

Another consequence of the Doxastic Consistency requirement is that the credibility function derived from an uninformative refinement is fully characterized by only a few terms of the initial credibility function. Let $(\Omega, \Re, Cr, EC_t^Y)$ be a belief state. Let R be an uninformative elmentary refinement from $\Re$ to $\Re'$. For $X\in\Re'$, define $A = R^{-1}(X)$ as the 'smallest' subset of $\Re$ such that its image under R contains X: $A = \cap_{B\in\Re,\, X\subseteq R(B)} B$. Let Cr' be the credibility function derived from Cr on $\Re'$ by R. Then Cr'(X) depends only on the terms Cr(A), $Cr(\overline{A})$

and Cr(Ω). In the illustrative example, $Cr_2(\omega_{21})$ would be the same if compute from Cr_1 defined on the algebra ℜ or from its coarsening defined on the algebra with atoms ω_{11} and $\omega_{12} \cup \omega_{02}$ (see theorem 3). Thus there are functions f_1, f_2 ... such that:

$$Cr_2(\omega_{21}) = f_1(Cr_1(\omega_{11}), Cr_1(\omega_{12} \cup \omega_{02}), Cr_1(\Omega))$$

With: $Cr_1(\omega_{11}) = f_2(Cr_0(\omega_{01}), Cr_0(\omega_{02}), Cr_0(\Omega))$

$Cr_1(\omega_{12} \cup \omega_{02}) = f_3(Cr_0(\omega_{01}), Cr_0(\omega_{02}), Cr_0(\Omega))$,

one has: $Cr_2(\omega_{21}) = f_4(Cr_0(\omega_{01}), Cr_0(\omega_{02}), Cr_0(\Omega))$.

Iterating the procedure through the set of appropriate elementary refinements, the property is extended to any refinement.

Proof: Let $(\Omega, \Re, Cr, EC_t^Y)$ be a belief state. Let R be an uninformative refinement from ℜ to ℜ'. For $X \in \Re'$, let $A = \cap_{B \in \Re,\, X \subseteq R(B)} B$. Let Cr' be the credibility function derived from Cr on ℜ' by R.

the value Cr'(X) of the credibility function Cr' induced from a credibility function Cr defined on ℜ by a refinement R from ℜ to ℜ' is fully defined from Cr(

This equality has the consequence that
$Cr_2(\omega_{21}) = g(\omega_{21}, Cr_0(\omega_{01}), Cr_0(\omega_{02}), Cr_0(\omega_{01} \cup \omega_{02}))$

Another consequence of requirement A4.2 is that $Cr_2(\omega_{21})$ is fully defined by $Cr_1(\omega_{11})$, $Cr_1(\omega_{12} \cup \omega_{02})$ and $Cr_1(\omega_{11} \cup \omega_{12} \cup \omega_{02})$.

Indeed, by requirement A4.1, Cr_1 is fully defined by $Cr_0(\omega_{01})$, $Cr_0(\omega_{02})$ and $Cr_0(\omega_{01} \cup \omega_{02})$, and $Cr_2(\omega_{21}) = Cr_2(\omega_{21})$

Cr1(

The impact of the uninformative refinement, i.e., the nature of the g function in requirement A4.1, is examined in the next section, simultaneously with the belief revision process. With a few extra requirement, we could find the mathematical structure of the g function introduced in requirement A4.1, but the proof is laborious (Smets, 1993c). As far as the conditioning process will have to be also analyzed and as the derivation of the g function is much simpler when both refinement and conditioning are studied simultaneously, the derivation of the g function will be deferred to the next section.

5. Belief Revision.

5.1. The conditioning process.

We consider now what happens to Cr when the evidential corpus EC_t^Y changes. We will not study all possible forms of changes of EC_t^Y. We restrict ourselves to the form of changes encountered in probability theory, i.e., conditioning. This form of belief revision results from the adjunction to EC_t^Y of a new piece of evidence assumed to be true (Gärdenfors, 1988). The only pieces of evidence considered in the revision process are those met classically in probability theory, those that only constraint the truth status of some propositions that belong to the credibility domain (called the ‘explicit conditions’ in Wang (1993)). We do not consider revision on propositions like: ‘the belief of proposition A is .7’ (Domotor, 1985) or ‘the principle of maximum entropy is applicable’. Besides the revision should neither be confused with updating (Katsuno and Mendelzon, 1992) nor with imaging, its probabilistic counterpart (Lewis, 1976, Gärdenfors, 1988). The evidential corpus is not *updated* in order to keep it up to date when the world described by it changes (Dubois and Prade, 1994b, Léa Sombe, 1994). It is *revised* by the adjunction of new information (Dubois and Prade, 1994a). Worlds do not change, only our knowledge about which is the actual world changes.

To illustrate the revision considered here, we continue with the failure diagnosis of example 3.

Example 3. Continuation 1. Generic Revision. You learn that none of the circuits made at factory F_B used in the failed equipment were painted pink. This piece of evidence is denoted Ev_1. Let EC_1 denote Your evidential corpus after Ev_1 has been added to EC_0. Under EC_1, $B =_{EC_1} W$, as knowing that the circuit has been made at factory F_B is now equivalent to knowing that the circuit is white. The impact of the conditioning information Ev_1 results in a transformation of the belief state $(\Omega, \Re', Cr', EC_0)$ into a new belief state $(\Omega, \Re', Cr_1, EC_1)$, i.e., Cr’ is transformed into a new credibility function Cr_1. Cr_1 must satisfy certain constraints in order to comply with Doxastic Consistency (requirement A.2.2).

Before learning Ev_1, we had: $A =_{EC_0} G$, $B =_{EC_0} W \cup P$, and Cr_0 was quantifying Your beliefs over $\Omega = \{A, B\}$. After learning Ev_1, we have: $X \cup P =_{EC_1} X$ for X being Ø, G, W or $G \cup W$. We also have $A =_{EC_1} G$, $B =_{EC_1} W$, and Cr_1 quantifies Your beliefs over $\Re'$. By Doxastic Consistency we have now:

$$Cr_1(G) = Cr_0(A), \quad Cr_1(P) = 0, \quad Cr_1(W) = Cr_0(B)$$
$$Cr_1(G \cup P) = Cr_0(A), \quad Cr_1(G \cup W) = Cr_0(A \cup B), \quad Cr_1(P \cup W) = Cr_0(B)$$
$$\text{and} \quad Cr_1(G \cup P \cup W) = Cr_0(A \cup B). \qquad (5.1)$$

□

Formally You is in an initial belief state (Ω, $\mathfrak{R}$, Cr, EC_0) where Cr quantifies Your beliefs at time t_0 about which subsets of worlds of Ω among those in $\mathfrak{R}$ include the actual world ω_0. Then at time $t_1>t_0$, You learn for sure that 'the actual world ω_0 is not in $\overline{A}$' for $A\in\mathfrak{R}$. We denote this information by Ev_A. We also suppose that, between t_0 and t_1, You have not learned anything relevant to Your knowledge about which world is the actual one. So Ev_A is the first information relevant to the actual world ω_0 obtained by You since t_0. Your evidential corpus at t_1 results from the revision of EC_0 by the information Ev_A. We denote the revised evidential corpus by EC_A, and the 'addition' is symbolized by $\oplus$, so $EC_A = EC_0 \oplus Ev_A$.

We assume that Ev_A is compatible with EC_0, i.e., that $A\cap[\![EC_0]\!]\neq\emptyset$. This is classically required in probability theory where the conditioning process is considered only for events with non zero probabilities.[5] The operator $\oplus$ corresponds to the expansion operator of Gärdenfors (1988).

The type of revising information considered in this paper is limited to those pieces of evidence that only say that ω_0 does not belong to some subset of Ω. In particular, in this section, we do not consider pieces of evidence that imply that other pieces of evidence already included in the evidential corpus must be eliminated from it (what we called a deconditionalization process, a process similar to a contraction process and analyzed in section 6 (Gärdenfors, 1988)), or partially discounted (as studied in Shafer, 1976, pages 251 et seq., Smets, 1993b).

The information Ev_A can be understood equivalently as: 'all worlds in Ω and not in A are accepted as impossible' or 'the actual world ω_0 is not in $\overline{A}$' or '$A =_{EC_A} \Omega$'. The revising evidence Ev_A is called the conditioning evidence and the particular revision process is called a conditioning process. Conditioning EC_0 on Ev_A is synonymous to adding Ev_A to EC_0. Beware we distinguish between 'the actual world ω_0 is not in $\overline{A}$' and 'the actual world ω_0 is in A'. The distinction is irrelevant if $\Omega = \Omega_L$ (see section 2.1:2). As we accept that Ω might be a strict subset of Ω_L, the second expression is stronger than the first. Indeed the second expression implies the first but it implies also that the credibility given to A should be maximal after revision, what is translated by the normalization process described in probability theory and in Shafer's work.

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[5] We could relax this compatibility assumption and consider cases where Ev_A is not compatible with EC_0. The result can be expressed in the TBM, but it is not very useful as it only gives a zero belief to every proposition (Smets, 1992a). Such a belief state represents a state of 'completely inconsistent belief'.

5.2. Markovian Revision.

In the belief state (Ω, $\Re$, Cr_0, EC_0), Cr_0 described Your belief given EC_0. How do You change Your beliefs given the addition of Ev_A to EC_0? Let Cr_A denote the credibility function (qualified as conditional) that results from the adjunction of Ev_A to EC_0.

It is assumed that Cr_A is completely determined from the credibility function Cr_0 and on Ev_A. This markovian property is classical. It reflects a lack of memory about how You obtained Your beliefs. All that counts in a belief state is where You stay for what concerns Your beliefs, not how You got there. Building a non markovian model leads either to the necessity to memorize all past beliefs, up to Your creation, or to use one-to-one transformation between Cr_0 and Cr_A (what is neither the case in probability theory nor in the TBM).

Requirement A5.1: Markovian Requirement. Let the belief state (Ω, $\Re$, Cr_0, EC_0). For $A \in \Re$, let Ev_A be the proposition $A =_{EC_A} \Omega$ where Ev_A be compatible with EC_0. Let (Ω, $\Re$, Cr_A, $EC_0 \oplus Ev_A$) be the belief state after Ev_A has been added to EC_0. It is assumed that Cr_A is completely determined by Cr_0 and A.

5.3. A second form of revision.

We reconsider the failure diagnosis of example 3 as presented in section 4.2. Instead of considering the generic revision, we consider another form of revision, called the factual revision.

Example 3. Continuation 2. Factual Revision. We are in the situation as described in example 3, section 4.2, so the revision information considered in continuation 1 is not taken in consideration. Instead**,** You possess a fully reliable sensor that is only able to detect if the color of a circuit is pink or not, so it cannot distinguish between green and white circuits. You learn that Your sensor has been applied to the broken circuit and has reported that the broken circuit is not pink. This new piece of evidence is denoted Ev_2. Let EC_2 be Your evidential corpus after Ev_2 has been added to EC_0. Under EC_2, $B =_{EC_2} W$, as knowing that the broken circuit has been made at factory F_B is now equivalent to knowing that the broken circuit is white. Let Cr_2 be the credibility function obtained after conditioning Cr' on Ev_2. By the Doxastic Consistency, Cr_2 satisfies relations similar to (5.1):

$Cr_2(G) = Cr_0(A)$, $\quad Cr_2(P) = 0$, $\quad Cr_2(W) = Cr_0(B)$

$Cr_2(G \cup P) = Cr_0(A)$, $\quad Cr_2(G \cup W) = Cr_0(A \cup B)$, $\quad Cr_2(P \cup W) = Cr_0(B)$

and $Cr_2(G \cup P \cup W) = Cr_0(A \cup B)$. (5.2)

□

The difference between the sets of doxastic equivalencies described in the two revisions resides in the fact that the second concerns only the broken circuit, whereas the first concerns all circuits made at factory F_B. *But as far as Your beliefs concern only the broken circuit, the two cases are equivalent for the problem You tries to solve.*

The two cases would be different if You had selected one circuit at random and bet on its color. In the case of generic revision, You would start with some probability that the circuit that will be selected has been made in F_A (in F_B). Then learning about the three colors, You would build a probability measure over the three colors. Finally, learning that all B-circuits are in fact white, You would reassess Your beliefs over the two remaining colors and obtain the same solution as we obtained (5.1). What You had built over the three colors was based on the assumption there were three colors, an assumption that turns out to be erroneous, and thus probabilities must be reassess from scratch, i.e., from Your state of belief You had before learning about the three colors.

In the case of factual revision, You would build the same probability measure over the three colors as in the previous case. Then You would learn that the selected circuit is not pink. You would condition Your beliefs over the two remaining colors through the Bayesian conditioning rule.

But these stories are not those we are considering. The generic revision solution is uncontroversial and will not be further discussed. In the factual revision case, we do not have any underlying random selection: there is a broken circuit and we learn information about it. For instance, does the information about the colors and the fact that the broken circuit happens not to be painted in the pink give any reason to modify Your belief that the broken circuit is an A-circuit? We don't think so. You had some reasons to believe that the broken circuit was an A-circuit, and the factual information should not change Your beliefs about it, i.e.,

$Cr_2(G) = Cr_0(A)$,

By a similar argument, we get the equalities (5.2).

The mathematical consequences of the equalities (5.2) are enormous. They almost imply the mathematical structures of both the conditioning and the uninformative refinement processes. We will get:

for the uninformative refinement process:

$Cr'(P) = Cr'(W) = 0$,

$Cr'(G \cup P) = Cr'(G \cup W) = Cr_0(A)$, etc...

for the conditioning process:

$Cr_2(G) = Cr'(G \cup P) - Cr'(P) = Cr_0(A)$

$Cr_2(W) = Cr'(W \cup P) - Cr'(P) = Cr_0(B)$, etc...

These are the solutions described in the TBM and in every Dempster-Shafer models (except for the normalization). In particular, the markovian requirement is satisfied by the conditioning process. It looks like Cr_2 depends on Cr_0, but this is achieved only through Cr', and Cr_2 depends only on Cr', as requested by requirement A5.1.

5.4. The meaning of 'credibility'.

The argument relative to the factual conditioning as developed here is central to understand where our theory departs from the classical probabilistic approach.

Consider the **medical diagnostic process**. Frequentists assume that the patient has been selected at random from the population of patients presenting the observed symptoms, an assumption usually void of any reality: the patient's presence does not result from any random selection. Bayesians claim that probabilities appear because they describe the clinician a priori opinion about the disease Your patient could be suffering from. From this a priori probability, other probabilities result after appropriate conditioning. This is the solution we would obtain in the TBM if such a priori probability was adequately representing the clinician's a priori opinion. But this is exactly the point we are not accepting. We claim that a priori opinions are usually not adequately represented by probability functions, arguing belief functions are more adequate, even though the idea of 'family of probability functions' might be another alternative (Walley, 1991, Voorbraak, 1993). The fact that the patient comes from a population where there are 999 cases with disease A and one without does not mean this proportion is relevant to the clinician's a priori belief about the fact the patient presents disease A. It would if the clinician knew the patient had indeed been selected at random in such a population. But we are studying the case where such a selection has not been used (or at least is not known by us to have been used). The credibility function we develop are quantifying the beliefs obatined in such general cases.

The measure of credibility we study is analogous to the one encountered in judiciary context when culpability has to be assessed. Consider **the rodeo paradox** where out of 1000 persons who attend it, only one paid the entrance fee, the others having forced the gate. Police does not know who paid. Police arrests one person who attended the rodeo. I am the judge to whom the policeman bring the arrested person who claims - of course - he is the one who paid. If I had to bet on Your culpability, I surely would bet with high probability on it, but this does not mean I accept that he is culprit. I would bet he did not pay (because almost nobody paid) but I have no reason whatsoever to believe that this person did pay or not (because no evidence is brought forward that would justify such a belief). This difference between betting and belief parallels the difference we introduce between the pignistic and the credal levels. The quantification we focus at represents the strength of 'good reasons' in the expression 'I have good reasons to believe'. In the TBM, we accordingly define Cr(A) as the amount of 'justified specific support' given to A (Smets and Kennes, 1994). Similar, if not identical, ideas explain the origin of the evidentiary value model (Gärdenfors et al., 1983).

The credibility we study is not unsimilar to **the concept of provability**, and it has even been suggested that the degree of belief that a proposition is true represents the probability of proving its truth (Pearl, 1988), except the revision processes are more subtle than the one considered here (Smets, 1991). Indeed the underlying probability measure introduces extra constraints that must be handled appropriately.

5.5. Iterated Conditioning.

To derive the impact of the conditioning process on the credibility function, we introduce the idea of iterated conditioning. Let EC_0 be the initial evidential corpus held by You at t. Let $A,B \in \Re$, and Ev_A (Ev_B) be the piece of evidence that states that the actual world is not in $\overline{A}$ ($\overline{B}$). Suppose You adds 1) Ev_A to EC_0, and then Ev_B to the revised evidential corpus, or 2) Ev_B to EC_0, and then Ev_A to the revised evidential corpus, or 3) directly $Ev_{A \cap B}$ to EC_0. The final evidential corpora are the same in the three cases, i.e., $(EC_0 \oplus Ev_A) \oplus Ev_B = (EC_0 \oplus Ev_B) \oplus Ev_A = EC_0 \oplus Ev_{A \cap B}$. This property is satisfied by the expansion process. It means that the three belief states obtained by these conditionings are the same, hence the order under which credibility function are conditioned is irrelevant. This is proved in Theorem 4.

Theorem 4: Let the belief state $(\Omega, \Re, Cr, EC_0)$. Let Ev_A, Ev_B and $Ev_{A \cap B}$ be the propositions $A =_{Ev_A} \Omega$, $B =_{Ev_B} \Omega$, $A \cap B =_{Ev_{A \cap B}} \Omega$, respectively. Then the three revision processes:

1) conditioning the credibility function Cr on Ev_A, and the result on Ev_B,
2) conditioning the credibility function Cr on Ev_B, and the result on Ev_A,
3) conditioning the credibility function Cr on $Ev_{A \cap B}$

induce the same conditional credibility function.

Proof. The three conditioning processes result in adding the same information in EC_0. The resulting conditional credibility functions are equal by Doxastic Consistency (requirement A2.2.) QED

We can also prove that the conditional credibility function $Cr_A(B)$[6] depends only on the beliefs obtained by coarsening $\Re$ into the algebra built on the three atoms $B \cap A$, $\overline{B} \cap A$, $\overline{A}$, and after conditioning, $Cr_A(B) = Cr_A(B \cap A)$ for any $A,B \in \Re$.

Theorem 5: Let the belief state $(\Omega, \Re, Cr, EC)$. For $A \in \Re$, let Ev_A be the proposition $A =_{Ev_A} \Omega$. Let $(\Omega, \Re, Cr_A, EC \oplus Ev_A)$ be the belief state obtained after conditioning the previous belief state on Ev_A. Then :

1: $Cr_A(B) = 0 \qquad \forall B \subseteq \overline{A}, B \in \Re$
2: $Cr_A(B) = Cr_A(B \cap A) \qquad \forall B \in \Re$

and 3: there is an f function such that $\forall B \in \Re$, $B \subseteq A$,

$$Cr_A(B) = f\,(Cr(B \cap A), Cr(\overline{B} \cap A), Cr(\overline{A}), Cr(A), Cr(B \cup \overline{A}), Cr(\overline{B} \cup \overline{A}), Cr(\Omega))$$

Proof. It results from requirement A2.2 and an appropriate coarsening of $\Re$ detailed in the appendix.

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[6] We use Cr_A to denote the conditional credibility function, but in the proof presented in the appendix, we will also use the notation Cr(.|A). Both notations are equivalent here.

5.6. Doxastic Stability.

The forthcoming theorem 6 formalizes the idea developed in example 3. An uninformative refinement of one atom ω of $\Re$ (factory f_B in example 3) into two sets of new atoms A and B in $\Re'$ (pink and white) followed by a conditioning on $\overline{B}$ (not pink) brings You back into the same belief state as You were before applying the refinement. Hence, the credibility function will stay ‘unchanged’ (except the algebras has changed). This requirement is called the requirement of Doxastic Stability, i.e., the stability of the belief state after eliminating some of the alternatives created by an uninformative refinement.

Requirement A5.2: Doxastic Stability.
Let the belief state $(\Omega, \Re, Cr, EC_0)$. Let R be an uninformative refinement from $\Re$ to $\Re'$. Let ω be an atom of $\Re$, and $R(\omega) = A \cup B$ where $A \cap B = \emptyset$, $A \neq \emptyset$, $B \neq \emptyset$. Let $Ev_{\overline{B}}$ be the piece of evidence that states that all atoms in B are impossible and let $EC_1 = EC_0 \oplus Ev_{\overline{B}}$, so $R(\overline{\omega}) \cup A =_{EC_1} \Omega$. Then under EC_1, $R(X) \cap \overline{B}$ and $R(X)$ are doxastically equivalent for every X in $\Re$: $R(X) \cap \overline{B} =_{EC_1} R(X)$.

5.7. Homomorphism and Preservation.

Gärdenfors (1988) suggests two compelling properties for probabilistic revision functions, the homomorphism and the preservation requirements. Homomorphism states that revision and convex combination commute. Homomorphism is not satisfied in probability theory because of the normalization. We first illustrate the meaning of the homomorphism requirement in the next continuation of the horse-race of example 1.

Example 1. Continuation 2. In the horse race example, suppose that You learn that Carol is a sure loser. You can derive the conditional credibility function either directly from the combined credibility function Cr_{12} or from the linear combination of the individual credibility functions Cr_1 and Cr_2. This requirement would have been satisfied in probability theory if probabilities had not been normalized. ◻

The homomorphism requirement corresponds to the case where Cr’ and Cr” represent Your beliefs on a belief domain $\Re$ in context C_1 and C_2, respectively. The context will be chosen at random (with chance α). Cr represents Your beliefs over $\Re$ before selecting the context and Cr is indeed a credibility function as was shown in section 3.

Requirement A5.3: Homomorphism: Given a propositional space $(\Omega, \Re)$ and three credibility functions Cr, Cr’ and Cr” defined on $\Re$ and based on the evidential corpus EC. Let $A \in \Re$ and let Cr_A, Cr'_A and Cr''_A be the conditional credibility functions induced by adding Ev_A to EC.

If $Cr = \alpha\, Cr' + (1-\alpha)\, Cr''$, $\alpha \in [0,1]$, then $Cr_A = \alpha\, Cr'_A + (1-\alpha)\, Cr''_A$.

The Preservation Requirement asserts essentially that a proposition as much believed as a tautology will be as believed as the conditioning proposition after conditioning. We illustrate what is meant by the preservation requirement.

Example 1. Continuation 3. Consider the horse race example involving four horses: Allan, Blues, Carol and Daisy. Suppose You learn that Daisy is a sure loser. Then {Allan, Blues, Carol} and{Allan, Blues, Carol, Daisy} are Doxastically Equivalent, hence Cr({Allan, Blues, Carol}) = Cr({Allan, Blues, Carol, Daisy}). Then if You also learn that Carol is a sure loser, then {Allan, Carol} and {Allan, Carol, Daisy} are Doxastically Equivalent, hence $Cr_{not-Carol}$({Allan, Blues, Carol}) = $Cr_{not-Carol}$({Allan, Blues, Daisy}). □

Requirement A5.4: Preservation: Given the credibility space $(\Omega,\Re,Cr)$,

if $Cr(B) = Cr(\Omega)$ for some $B\in\Re$, then $Cr_A(B) = Cr_A(A)$ for all $A\in\Re$.

In order to follow Gärdenfors' initial presentation of the preservation principle, we should add the hypothesis $Cr(\bar{A})<Cr(\Omega)$. It can be relaxed in the present context as we do not normalize, (hence no division is involved). If it happened that $Cr(\bar{A})=Cr(\Omega)$, then we would get $Cr_A(B)=0\ \forall B\in\Re$, in which case $Cr_A(\Omega)=0$, a belief that describes a state of complete contradiction not dissimilar to the one encountered in logic when You simultaneously know something and its contrary. This problem is studied in Smets (1992a).

We are now ready to prove the theorem that states the mathematical structure of both the uninformative refinement and the conditioning process. They turn out to be those described for belief functions. In particular, the conditioning rule is Dempster's rule of conditioning (except for the normalisation factor).

Theorem 7: Let the belief state $(\Omega, \Re, Cr, EC)$. Assume the requirements A2, A3, A4, A5.1 to A5.4. Let R be an uninformative refinement from $\Re$ to $\Re'$ and Cr' be the credibility function derived from Cr on $\Re'$ by R. Then:

$$Cr'(X) = \max_{Y:R(Y)\subseteq X} Cr(Y) \quad \text{for all X in } \Re'. \tag{5.3}$$

For $A\in\Re$, let Ev_A be the conditioning information $A =_{Ev_A} \Omega$ and Cr_A be the conditional credibility function obtained from Cr after adding Ev_A to EC. Then:

$$Cr_A(B) = Cr(B\cup\bar{A}) - Cr(\bar{A}) \qquad \text{for all } B\in\Re. \tag{5.4}$$

** 5.8. Factual conditioning and deductive logic.

The way we treat the factual conditioning is at the core of our modelization, and deserves further explanation as it clashes with the a priori opinions of the classical bayesian approach. We are going to reanalize the broken circuit example, using predicate logic notation. We will show that our factual conditioning is related to deduction, whereas the Bayesian approach is related to abduction.

1.1 $\forall x: C(x) \supset ((A(x)\wedge\neg B(x)) \vee (\neg A(x)\wedge B(x)))$ Circuits are either from F_A or from F_B.

1.2	$\exists!x: C(x) \wedge Brk(x)$	One circuit is Broken.
1.3	$C(\alpha) \wedge Brk(\alpha)$	Its name is α.
2.1	$\forall x: A(x) \supset G(x)$	F_A circuits are Green.
2.2	$\forall x: B(x) \supset ((W(x) \wedge \neg P(x)) \vee (\neg W(x) \wedge P(x)))$	F_B circuits are either White or Pink.
Gen	$\forall x: C(x) \wedge B(x) \supset \neg P(x)$	None of the F_B circuits are Pink.
Fact	$\neg P(\alpha)$	The broken circuit is not Pink

The initial step of example 3 (section 4.5) is summarized by expressions 1.1 to 1.3 that formalize EC_0. Based on this knowledge, You build the credibility function Cr_0 on the origin of the broken circuit. The terms $Cr_0(A(\alpha))$ and $Cr_0(B(\alpha))$ denote the strength of Your belief that the broken circuit comes from factory F_A and F_B, respectively. For simplicity sake, we use the grounded propositions as arguments of the credibility functions, their relation to a set of world being immediate.

The next information is the uninformative refinement described by expressions 2.1 and 2.2. Expressions 1.1 and 2.1 imply that $\forall x: A(x) \equiv G(x)$. Identically expressions 1.1 and 2.2 imply that $\forall x: B(x) \equiv ((W(x) \wedge \neg P(x)) \vee (\neg W(x) \wedge P(x)))$. Let Cr' denote Your belief about the color of the broken circuit. Given these two logical equivalences, Cr' must satisfy: $Cr'(G(\alpha)) = Cr_0(A(\alpha))$, $Cr'(W(\alpha) \vee P(\alpha)) = Cr_0(B(\alpha))$. Some values of Cr', like $Cr'(W(\alpha))$, cannot not be deduced form the Doxastic Consistency. They will be obtained in theorem 7.

The generic revision (section 5.1, example 3, continuation 1) results from the information given by the expression Gen. Expressions 2.2 and Gen imply that $\forall x: B(x) \equiv W(x)$. Let Cr_1 denote Your belief about the color of the broken circuit given expressions 1.1 to 1.3, 2.1, 2.2 and Gen. Cr_1 satisfy: $Cr_1(G(\alpha)) = Cr_0(A(\alpha))$, $Cr_1(W(\alpha)) = Cr_0(B(\alpha))$, etc... (see 5.1).

The factual revision ((section 5.3, example 3, continuation 2) results from the information given by the expression Fact. Expressions 2.2 and Fact imply that $B(\alpha) \equiv W(\alpha)$. Let Cr_2 denote the credibility function that represents Your belief given expressions 1.1 to 1.3, 2.1, 2.2 and Fact. In section 5.3, we had assumed that Cr_2 satisfy also $Cr_2(G(\alpha)) = Cr_0(A(\alpha))$, $Cr_2(W(\alpha)) = Cr_0(B(\alpha))$, etc... (see 5.2). It was based on the logical equivalences $A(\alpha) \equiv G(\alpha)$ and $B(\alpha) \equiv W(\alpha)$. Some critics might feel that these equalities are inappropriate and should be replaced by inequalities: $Cr_2(G(\alpha)) > Cr_0(A(\alpha))$, $Cr_2(W(\alpha)) < Cr_0(B(\alpha))$... The origin of the divergence can be found in the difference between deduction and abduction.

These two schemes are:
1) abduction schema: observing the consequent of an implication increases the support that the antecedent holds.
2) deduction schema: observing the consequent of an implication does not tell anything about the fact that the antecedent holds.

In the present context, these two schemes become:

1) abduction schema: from $\forall x$: $A(x) \supset \neg P(x)$ and $\neg P(\alpha)$, Your belief that the broken circuit α was made in Factory F_A increased, hence $Cr_2(G(\alpha)) > Cr_0(A(\alpha))$.
2) deduction schema: from $\forall x$: $A(x) \supset \neg P(x)$ and $\neg P(\alpha)$, You can deduce nothing on $A(\alpha)$, hence the belief that the broken circuit α was made in Factory F_A is unchanged, hence $Cr_2(G(\alpha)) = Cr_0(A(\alpha))$.

In a certain sense, our model to represent quantified beliefs is an extension of deductive logic, not of abductive logic.

For sake of completeness, we propose the following scenario to represent the case where a circuit is randomly selected, and then we learn it is not pink. We accept that every circuit has the same chance of being selected, adapting the story to cope with non equal selection chances is immediate.

1.1	$\forall x: C(x) \supset ((A(x) \wedge \neg B(x)) \vee (\neg A(x) \wedge B(x)))$	Circuits are either from F_A or from F_B.
1.2	$\exists! x: C(x) \wedge Sel(x)$	One circuit is Selected.
1.3	$C(\alpha) \wedge Sel(\alpha)$	Its name is α.
2.1	$\forall x: A(x) \supset G(x)$	F_A circuits are Green.
2.2	$\forall x: B(x) \supset ((W(x) \wedge \neg P(x)) \vee (\neg W(x) \wedge P(x)))$	F_B circuits are either White or Pink.
Fact	$\neg P(\alpha)$	The selected circuit is not Pink

Let $|A|$, $|W|$... denote the number of distinct circuits in the equipment that were made at Factory F_A, that were painted White..., respectively. By assumption, Your belief that the selected circuit α was made at F_A is $\frac{|A|}{|A| + |B|}$ (accepting Hacking frequency principle (1965) that numerically equates belief and chance). The information about the color (2.1 and 2.2) implies that Your belief that the color of the selected circuit is Green is $\frac{|G|}{|G| + |W| + |P|}$, what is of course equal to $\frac{|A|}{|A| + |B|}$. After learning that the selected circuit was not pink (Fact), Your belief that the color of the selected circuit is Green becomes $\frac{|G|}{|G| + |W|}$. Once $|P|>0$, an acceptable fact, this new belief is larger than the belief You had before learning Fact, just as with the abduction schema.

The reason why the Bayesian schema clashes with our approach of the factual revision lays in the necessity to feed into the Bayesian model an a priori belief for each circuit that expresses Your a priori belief that this circuit is the broken circuit. Besides, that a priori belief must be represented by a probability function (something like the equi a priori chance). It happens that probability functions cannot represent states of partial or total ignorance as we need in fact. In our analyzis, we never used the information about the numbers of F_A, W... circuits (nor about some measure of their corresponding sets).

** 5.8. Convex Capacities.

So far we have derived what are the impacts of uninformative refinement and conditioning on a credibility function, but Cr is not even a capacity monotone of order 2 (called convex capacities, Choquet 1953, Chateauneuf and Jaffray, 1989). Convex capacities could be justified by assuming that conditioning does not decrease relevant credibilities. Consider $Cr_A(C)$ and $Cr(C)$ for $A,C\in\Re$, $C\subseteq A$. You had some belief Cr(C) that $\omega_0\in C$. Then You learns Ev_A, i.e., that ω_0 is not in $\bar{A}$. So some of the worlds You had considered potentially believable are to be rejected as impossible by Ev_A. It seems that Your values of the revised belief for C should not decrease. Why eliminating some atoms not in C (those in $\bar{A}$) should decrease Your belief that ω_0 is in C? Therefore we propose requirement A5.5 that generalizes that idea. That requirement implies that Cr is a convex capacity.

Requirement A5.5: Convex capacities.
Let $(\Omega, \Re, Cr)$ be a credibility space. For $A,B\in\Re$, let Cr_A and $Cr_{A\cap B}$ be the conditional credibility functions induced from Cr by the evidence Ev_A and $Ev_{A\cap B}$. Then:

$$\forall B\in\Re, C\subseteq A\cap B \qquad Cr_{A\cap B}(C) \geq Cr_A(C).$$

Theorem 8. Let $(\Omega, \Re, Cr)$ be a credibility space where Cr satisfies requirements A2, A3, A4 and A5.1 to A5.5. Then Cr is a convex capacity.

In fact, requirement A5.5 is not necessary for our task. It is given just to show how we can show that Cr is a convex capacity. We shall show in section 6 that Cr is in fact a belief function, a property that implies that Cr is a convex capacity, without using requirement A5.5.

** 5.9. Why are probability functions and plausibility functions inadequate?

Before proving that all credibility functions are belief functions, we consider why probability functions and plausibility functions are inadequate to represent quantified beliefs.

To show that probability functions are not adequate, we consider the problem of iterated uninformative refinements. As an illustrative example, take $\Omega_0 = \{a, b\}$, $\Omega_1 = \{a, b_1, b_2\}$, and $\Omega_2 = \{a, b_1, b_{21}, b_{22}\}$. Let R_1 be a refinement from $(\Omega_0, 2^{\Omega_0})$ to $(\Omega_1, 2^{\Omega_1})$ such that $R_1(a) = \{a\}$, and $R_1(b) = \{b_1, b_2\}$. Let R_2 be a refinement from $(\Omega_1, 2^{\Omega_1})$ to $(\Omega_2, 2^{\Omega_2})$ such that $R_2(a) = \{a\}$, $R_2(b_1) = \{b_1\}$ and $R_2(b_2) = \{b_{21}, b_{22}\}$.

Let the belief state $(\Omega_0, 2^{\Omega_0}, Cr_0, EC_0)$. Let Cr_1 (Cr_2) be the credibility function induced from Cr_0 (Cr_1) on 2^{Ω_1} (2^{Ω_2}) by the uninformative refinement R_1 (R_2).

Consider the refinement R_{12} from $(\Omega, 2^{\Omega})$ to $(\Omega_2, 2^{\Omega_2})$ such that $R_{12}(a) = \{a\}$, $R_{12}(b) = \{b_1, b_{21}, b_{22}\}$, and let Cr_{12} be the credibility function induced from Cr_0 on 2^{Ω_2} by the uninformative refinement R_{12}. R_{12} is nothing but the result of combining R_1 with R_2. By the Doxastic Consistency Requirement, $Cr_2 = Cr_{12}$.

In order to achieve such an equality in probability theory, we need to know how $Cr_0(b)$ is distributed among b_1 and b_2, and how $Cr_1(b_2)$ is distributed among b_{21} and b_{22}. For one, that knowledge contradicts the Markovian Requirement that states that Cr_1 should depend only on Cr_0 and R_1, not on some extra information like the distributions of $Cr_0(b)$ between b_1 and b_2. The Markovian Requirement can only be satisfied if $Cr_0(b)$ is equally distributed between b_1 and b_2, in which case $Cr_1(b_2)$ should also be equally distributed between b_{21} and b_{22}. Thus $Cr_2(b_{21})$ would be equal to $Cr_0(b)/4$. The same rule applied to Cr_{12}, using R_{12}, implies that $Cr_{12}(b_{21}) = Cr_0(b)/3$, hence $Cr_{12} \neq Cr_2$, an inequality that contradicts the Doxastic Consistency Requirement. Hence equirepartition cannot de defended. This implies that probability functions are not fitted to represent beliefs once iterated uninformative refinements are introduced.

The Preservation Requirement is not satisfied by plausibility functions, the dual of the belief functions. This rejection seems adequate. We feel that Cr should represent the strength of belief and should behave like the modality used to represent categorical beliefs (the 'box' operator encountered in doxastic logic). Using plausibility functions to represent quantified beliefs would be equivalent to representing categorical beliefs by the 'diamond' operator. Of course, such an interpretation of 'belief' could be defended. The question is in defining what is meant by beliefs: we follow the classical doxastic logic interpretation (Hintikka, 1962).

In conclusion, probability functions are not expressive enough to satisfy our requirements, and plausibility functions do not cover our interpretation of the belief modality.

6. Credibility functions and belief functions.

6.1. Belief functions are credibility functions.

We want to show that the set of credibility functions is the set of belief functions. We first show that every belief function satisfies requirements A2, A3, A4 and A5. The reverse theorem requires the introduction of the concept of deconditionalization.

Theorem 9: Every belief function satisfies requirements A2, A3, A4 and A5.

Proof: see appendix.

6.2. Deconditionalization.

The aim of this paper is to determine the mathematical structure of the credibility functions. We now prove they are belief functions (monotone capacities of order infinite) by studying the concept of deconditionalization, i.e., the inverse of the conditioning process. Let the belief state $(\Omega, \mathfrak{R}, Cr, EC)$. Let Cr_X be the credibility function defined on $\mathfrak{R}$ after conditioning Cr on the evidence Ev_X for $X \in \mathfrak{R}$ that means $X =_{Ev_X} \Omega$. Suppose You learn that conditioning on

Ev_X was inappropriate, i.e., that all the reasons that lead You to condition Cr on Ev_X were unjustified. You want to erase the impact of Ev_X from Cr_X, and rebuild the credibility function Cr from which Cr_X had been obtained by its conditioning on Ev_X. This process is a special form of contraction (Gärdenfors, 1988). We call it a deconditionalization of Cr_X for Ev_X.

If You had memorized the value of Cr before its conditioning on Ev_X, the deconditioning process would be trivial: the result would be Cr. But because of the markovian requirement A5.1, such a memorization is absent, and Cr_X is all what You know when You must deconditionalize it. The memory of Cr is lost and the transformation (relation 5.4) between Cr and Cr_X that reflects the impact of Ev_X is not one-to-one, but many-to-one, so knowing Cr_X is not sufficient to recover Cr.

Formally, let $(\Omega, \mathfrak{R})$ be a propositional space. Let *Cr* be the set of credibility functions defined on $\mathfrak{R}$. For $X\in\mathfrak{R}$, let Cr_X be the set of conditional credibility functions obtained by conditioning the elements of *Cr* on Ev_X by (5.4).

The impact of conditioning the elements of *Cr* on Ev_X can be described by an operator $S_X : \mathit{Cr}\rightarrow\mathit{Cr}_X$ such that:

$$Cr_X = S_X \circ Cr \qquad \text{for all } Cr\in\mathit{Cr} \tag{6.1}$$

By theorem 4, iterating conditioning on Ev_X and Ev_Z is equivalent to directly conditioning on $Ev_{X\cap Z}$. Hence the conditioning operator satisfies for all $X,Z\in\mathfrak{R}$:

$$S_X \circ S_Z = S_{X\cap Z} \tag{6.2}$$

Consider now the deconditioning operators for Ev_X, i.e., the operator that maps Cr_X into *Cr*. Let $S_{\bar{X}}$ be such an operator. If conditioning had been one-to-one, $S_{\bar{X}}$ would just be the inverse of S_X, but given S_X is many-to-one, $S_{\bar{X}}$ is a generalized inverse. $S_{\bar{X}}$ must satisfy:

$$S_X \circ S_{\bar{X}} \circ S_X = S_X \tag{6.3}$$

and

$$S_{\bar{X}} \circ S_{\bar{X}} = S_{\bar{X}} \tag{6.4}$$

Indeed re-conditioning after deconditioning annihilates the effect of the deconditioning (6.3) and deconditioning twice has the same impact as deconditioning once (6.4), just like conditioning twice on the same piece of evidence was equal to conditioning only once on that piece of evidence: $S_X \circ S_X = S_X$.

Even though S_X is unique, there are many operators $S_{\bar{X}}$ satisfying (6.3) and (6.4)[7]. Let $\mathfrak{S}_{\bar{X}}$ be the set of deconditioning operators $S_{\bar{X}}$ satisfying (6.3) and (6.4).

Example 4: In order to explain the origin of the next requirement, suppose that Cr_X quantifies Your beliefs over $\mathfrak{R}$ based on an evidential corpus EC that contains the conditioning evidence Ev_X for $X\in\mathfrak{R}$. You then learn that the evidence Ev_X was unjustified and its impact must be suppressed. What operator $S_{\bar{X}}$ will You use? Suppose another agent You* has some

[7] S_X and $S_{\bar{X}}$ are linear operators and can be written under matricial notation. Then $S_{\bar{X}}$ is a generalized inverse of S_X and both S_X and $S_{\bar{X}}$ are idempotent (Klawonn and Smets, 1992).

opinion about which operator $S_{\bar{X}} \in \mathfrak{S}_{\bar{X}}$ is to be used by You. You*'s opinion is represented by a credibility over $\mathfrak{S}_{\bar{X}}$. Suppose You* is sure about which $S_{\bar{X}} \in \mathfrak{S}_{\bar{X}}$ should be used by You to decondition Cr_X. Suppose You had no a priori about which operator is appropriate. You trust in You*. So You accept You*'s opinion that the appropriate operator is $S_{\bar{X}}*$ and You use this $S_{\bar{X}}*$ to decondition Cr_X. Of course, the result must be a credibility function over $\Re$. We want that the choice of $S_{\bar{X}}*$ by You* can be made independently of the value Cr_X representing Your belief over $\Re$. Thus, for every Cr_X in Cr_X and every $S_{\bar{X}} \in \mathfrak{S}_{\bar{X}}$, $S_{\bar{X}} \circ Cr_X$ must be a credibility function. This constraint is sufficient to prove that the credibility functions are belief functions. □

The next requirement just formalizes the requirement detailed in example 4.

Requirement A6.1. Let $(\Omega, \Re)$ be a propositional space. Let Cr be the set of credibility functions defined on $\Re$. For $X \in \Re$, let Cr_X be the set of conditional credibility functions defined on $\Re$ after conditioning the credibility functions in Cr on the evidence Ev_X. Let $\mathfrak{S}_{\bar{X}}$ be the set of operators deconditioning the elements of Cr_X on Ev_X. For every $S_{\bar{X}}$ in $\mathfrak{S}_{\bar{X}}$ and every Cr_X in Cr_X, one has: $S_{\bar{X}} \circ Cr_X \in \mathit{Cr}$.

This requirement means that the set of operators for deconditioning on an evidence Ev_X does not depend of the credibility function to which they are applied. We want that if $S_{\bar{X}}$ is a deconditionalization operator so that it produces a credibility function when applied to some $Cr_X \in \mathit{Cr}_X$, than it produces a credibility function when applied to any $Cr_X \in \mathit{Cr}_X$. This requirement is sufficient to prove that credibility function are belief functions. Details are given in appendix 3.

Theorem 10: Every function that satisfies requirements A2, A3, A4, A5.1 to A5.4 and A6.1 is a belief function.

This concludes our task.

** 6.3. Other proofs that Cr is a belief function.

6.3.1) The proof that Cr is a belief function can be achieved differently. In particular, section 6.2's results can be derived by using the inverses of the coarsening operators instead of the inverses of the conditioning operators. We feel deconditioning is a clearer concept that de-coarsening, hence our choice.

We present several other sets of requirements that prove that credibility functions are belief functions. Unfortunately for most of them we cannot provide some definitive argument based on the primitive concept of credibility that would justify their acceptation. They only enlighten the danger incurred if Cr is a not a belief function.

6.3.2) In Smets (1993c) we show that credibility functions are belief functions by postulating the following closure property.

Let C_r be the set of credibility functions defined on an algebra $\Re$ with r atoms and that satisfy all the properties developed up till section 5. Let $\Re$' be a refinement obtained from $\Re$ by refining only one atom of $\Re$ into two new atoms. Let $Ext(C_r)$ be the set of credibility functions on $\Re$' that can be obtained by the application of such refinement operators from the credibility functions in C_r. Let $Clos(Ext(C_r))$ be the closure of $Ext(C_r)$ that contains all credibility functions that can be obtained from those in $Ext(C_r)$ through conditioning and convex combinations. In Smets (1993c), we postulated:

Requirement A6.2. Closure Property. $C_{r+1} = Clos(Ext(C_r))$.

I.e., any credibility function in C_{r+1} could be derived from some credibility functions in C_r through refinement, conditioning and convex combinations. As far as the Möbius transform of a credibility function defined on a frame of discernment with one atom is always non negative, and the considered transformations preserve the non negativity of the Möbius transform, credibility functions are thus belief functions.

6.3.3) Requirement A6.3. Let $(\Omega, \Re, Cr)$ be a credibility space. For $X,Y \in \Re$, $X \subseteq Y$, let Cr_X and Cr_{You} be the conditional credibility functions derived from Cr after conditioning on Ev_X and Ev_Y, respectively. Let v, v_X and v_{You} be the Möbius transforms of Cr, Cr_X and Cr_Y, respectively. Then,

A6.3a $v_X(X) \geq v_Y(Y)$.

A6.3b $v_X(A) \geq v_Y(A)$, $A \subseteq X$, $A \in \Re$.

Both requirements imply that $v(X) \geq 0$ for all $X \in \Re$, hence Cr is a belief function. But the Möbius transform has no natural interpretation so far. The meaning of v appears only once Cr is a belief function in which case, v(A) for $A \in \Re$ is the part of belief that supports that the actual world is in A without supporting the fact it belongs to any strict subset of A (Smets and Kennes, 1994). Requirements A6.3 are only technical and are not useful to justify that credibility functions are belief functions.

6.3.4) Equivalently, one can introduce the concept of a commonality function, a very useful mathematical object in the TBM. Given a credibility function and its Möbius transform v, the commonality function q on $\Re$ is defined by:

$$q(A) = \sum_{B: B \subseteq A} v(B).$$

Requirement A6.4. Let $(\Omega, \Re, Cr)$ be a credibility space. For $X,Y \in \Re$, $X \subseteq Y$, $q(X) \geq q(Y)$.

Requirement A6.4 is equivalent to requirement A6.3.a as $q(X) = v_X(X)$ for $X \in \Re$. In the TBM, q(X) is that part of belief kept uncommitted in a context where Ev_X is accepted. But just as with the Möbius transforms, the commonality functions have no natural interpretation before Cr is accepted as a belief function.

6.3.5) Shafer (1976) assumes that any measure of belief Cr on algebra $\Re$ should satisfy the following inequalities:

$\forall n \geq 1,\ A_1, A_2, \ldots A_n \in \Re,$

$$Cr(A_1 \cup A_2 \cup \ldots A_n) \geq \sum_i Cr(A_i) - \sum_{i>j} Cr(A_i \cap A_j) \ldots . -(-1)^n Cr(A_1 \cap A_2 \cap \ldots A_n),$$

what means of course that Cr is a belief function. In the TBM (Smets and Kennes, 1994), we assume the existence of parts of beliefs that support a proposition without supporting more specific propositions. These parts of belief are in fact the values of the Möbius transform of the belief function. Both approaches are strictly equivalent. We introduced the second in response to the criticism that the inequalities of Shafer were too artificial and difficult to accept as natural requirements for a measure of belief, hoping ours would be more ‘palatable’.

6.3.6) It can be proved that the following inequalities among conditional credibility functions are satisfied iff Cr is a belief function.

Requirement A6.5. Let $(\Omega, \Re, Cr)$ be a credibility space. For any n, any $A_1, A_2, \ldots A_n \in \Re$, and any $X \subseteq \cap_i A_i$

$$\sum_{I \subseteq \{1,2 \ldots n\}} (-1)^{n-|I|}\ Cr_{\cap_{i \in I} A_i}(X) \geq 0.$$

But how can these inequalities by justified? They are even worse than those initially defended by Shafer (1976).

6.3.7) Dubois and Prade (1986) have introduced the idea of complementary belief functions. Given a credibility function Cr on $\Re$ with v its Möbius transform and q its commonality function, they propose to define the set-function Cr* also defined on $\Re$ such that its Möbius transform v* satisfies $v^*(A) = v(\overline{A})$ for all $A \in \Re$. Then $Cr^*(A) + m^*(\emptyset) = q(\overline{A})$ and $Cr(A) + m(\emptyset) = q^*(\overline{A})$ where q* is the commonality function computed from Cr*. In order that Cr* be a credibility function, Cr must be a belief function. If it were not the case, it is always possible to generate another credibility function by coarsenings and refinements such that its complementary function would not be a credibility function as some of its values would be negative. The problem with such a justification is that the transformation Cr* does not have a natural interpretation unless the Möbius transform gets one, what is not the case as Cr is not yet proved to be a belief function.

6.3.8) Wong et al. (1990) have presented an axiomatic justification based on the representation of a belief-order relation ≥ (>) where B≥C (B>C) means ‘B is not less believed than C' ('B is more believed than C'). They replace the disjoint union requirement assumed in probability theory (Koopman 1940, Fine, 1973):

$A\cap(B\cup C)=\emptyset \Rightarrow (B\geq C \Leftrightarrow A\cup B\geq A\cup C)$

by a less restrictive requirement:

$C\subseteq B,\ A\cap B=\emptyset \Rightarrow (B>C \Rightarrow A\cup B\geq A\cup C)$.

Under this last requirement, the ≥ belief-ordering can always be represented by a belief function. Unfortunately, other functions like the convex capacities can also represent the ≥ ordering. The fact Wang et al. could not prove that the ≥ belief-ordering can *only* be represented by belief functions prompted us into developing the present axiomatization.

We hope future work might show that some of the alternative requirements can find a natural justification. This hope explains why we present these properties.

7. Conclusions.

We conclude this paper by first summarizing the major results (table 1), then answering several questions that could arise from our presentation.

1) We have been able to show under which conditions credibility functions are belief functions. We show that the set of credibility functions is a convex set, which was essentially achieved by accepting the existence of a chance setup, i.e. of objective probabilities. The major requirement is the doxastic stability that must be satisfied after eliminating some of the alternatives created by an uninformative refinement (requirement A5.2), the homomorphism (requirement A5.3) and preservation requirements for conditioning (requirement A5.4). Finally, we assume that the domain of the deconditionalization operators does not depend on the credibility functions to which deconditionalization is applied (requirement A6.1). Together these requirements imply that credibility functions are belief functions.

It could be argued that the maybe large number of requirements might be misleading, and in any case might reduce the confidence one has in their adequacy. It should be noted that we have derived many properties: the convex combination, the uninformative coarsening, the uninformative refinement, the conditioning and the deconditioning.

In probability theory, about the same number of requirements would be needed to derive the same set of properties. An enterprise similar to ours to justify probability theory would require something like the Koopman set for qualitative probabilities (Koopman, 1940) and their extensions for conditional probabilities (Fine, 1973). Besides the concept of uninformative refinement in not obvious within the probabilistic framework. So the critic that there are "too many requirements" is either not acceptable or should be applied identically to probability theory!

Definition.	Cr definition.		
A2.1:	*	Existence.	Cr is pointwise.
	*	Domain.	$[\alpha_\perp, \alpha_T]$
	*	Monotonicity.	If $A \subseteq B$, then $Cr(A) \leq Cr(B)$
	*	Lower limit.	$Cr(\emptyset) = \alpha_\perp$
A2.2:	*	Doxastic consistency.	If $A_1 =_{EC} A_2$, then $Cr_1(A_1) = Cr_2(A_2)$
Convexity.	$Cr_{12} = \alpha .Cr_1 + (1-\alpha).Cr_2$		
A3.1:	*	Compositionality.	$Cr_{12} = F_\alpha(Cr_1 , Cr_2)$
A3.2:	*	Continuity.	$F_\alpha(x,y)$ continuous.
A3.3:	*	Strict monotonicity.	$F_\alpha(x,y)$ strictly monotone.
A3.4:	*	Idempotency.	$F_\alpha(x,x) = x$
A3.5:	*	Probability Functions.	Probability functions are credibility functions.
Uninformative Coarsening			
A4.1:	*	Cr” = h(Cr, C)	
Uninformative Refinement			
A4.1:		Cr' = g(Cr, R)	
Conditioning			
A5.1:	*	Markovian Requirement.	Cr_A depends only on Cr and A.
A5.2:		Doxastic Stability.	Eliminating atoms built from an uninformative refinement.
A5.3:		Homomorphism.	Conditioning and convex combination commute.
A5.4:	*	Preservation.	If $Cr(B) = Cr(\Omega)$, then $Cr_A(B) = Cr_A(A)$.
Deconditioning			
A6.1.		Free deconditioning	$\forall S_{\bar{X}} \in \mathfrak{S}_{\bar{X}}, \forall Cr_X \in \mathit{Cr}_X,\ S_{\bar{X}} \circ Cr_X \in \mathit{Cr}.$

Table 1: List of requirements, their numbers, names and major properties. * indicates those requirements satisfied in probability theory.

It might also be worth emphasizing that most requirements are satisfied by probability functions (those requirements indicated by * in table 1).

2) **Decision making.** In section 1, we mention the existence of a credal and a pignistic level. In Smets (1990b) and Smets and Kennes (1994), we develop and justify the so called pignistic transformation that permits the construction of the probability function needed at the pignistic level for decision making from the credibility function held at the credal level. This construction protects the TBM against any criticism based on Static Dutch Book. In Smets (1993a) we also show how Diachronic Dutch Books are also avoided.

3) **Combining belief functions.** Classically, a major component of the models for quantified beliefs based on belief functions is the rule of combination by which two distinct pieces of evidence are combined in order to build a new belief function that reflects the impact of both pieces of evidence.

The conjunctive rule of combination, called the Dempster's rule of combination, has been proposed by Shafer (1976). He justifies it by assuming the initial Dempster model based on an underlying probability function, a one-to-many mapping and some stochastic independence (Shafer 1984, Shafer and Tversky, 1985). Later on, Smets (1990a), Klawonn and Schwecke (1992) and Hajek (1992) have presented justifications within the TBM where no underlying probability measure is assumed. They assume the same conditioning rule as derived in section 5 and justify Dempster's rule of combination by symmetry and associativity arguments. In Klawonn and Smets (1992), both the conditioning and the combination were derived by postulating the least commitment principle and the fact that the revision of a belief function results from a specialization process, i.e., a flow of the basic belief masses to their subsets.

The disjunctive rule of combination and the generalization of Bayes theorem to belief functions was introduced in Smets (1978, 1981), and was fully justified in Smets (1993b).

4) It might surprise that our credibility functions are not normalized. Indeed, we neither assume that $Cr(\Omega) = 1$, nor that $Cr_A(\Omega) = Cr_B(\Omega)$. The first case reflects the fact that Ω might be different from Ω_L (see section 2.2). The impact of such a difference is that the degree of belief given by You at t to Ω can be smaller than the degree of belief that could have been given to Ω_L. Nevertheless the difference $1-Cr(\Omega)$ is still better understood once conditioning is introduced. Suppose $Cr(\Omega) = 1$, the difference $1-Cr_A(\Omega)$ quantifies then the (partial) inconsistency present in $EC \oplus Ev_A$ (Smets, 1992a). The difference contains the belief that was given initially to a set $\bar{A}$ that turns out to be impossible given the revising information. Given Your initial evidential corpus EC, You had given some belief to $\bar{A}$. You then learns that $\bar{A}$ is impossible. This new piece of evidence Ev_A is partially contradictory with EC as it says that the belief that was given to $\bar{A}$ was inappropriate. The larger that belief, the larger the contradiction, the larger $1-Cr_A(\Omega)$. The fact that the amounts of contradiction between EC and Ev_A or Ev_B can be different explains why $Cr_A(\Omega)$ and $Cr_B(\Omega)$ can be different. As far as any credibility function result from the conditioning on EC, the difference $1-Cr(\Omega)$ that reflects the difference between Ω and Ω_L can be explained in the same way as a measure of internal inconsistency, of internal contradiction present initially in EC, i.e., when building Cr on $\Re$.

In conclusion, this paper has shown that under quite general conditions, the measures of quantified beliefs are belief functions. It provides a first detailed axiomatization that justifies the use of belief functions. It leads to the TBM described in Smets and Kennes (1994), a model for representing quantified beliefs that we feel more appropriate than the Bayesian model that restricts itself to the use of probability functions. Its use for statistical inference will be studied. This papers provides the justification of the model, a justification required before it can be applied to practical problems.

APPENDIX.

Appendix 1: The Choquet Capacities.

Let Ω be a set and let $\Re$ be a Boolean algebra of subsets of Ω. A (Choquet) monotone capacity of order n for $n \geq 2$ is a function C from $\Re$ to [0, 1] such that

1) $C(\emptyset) = 0$
2) $\forall A_1, A_2, \ldots A_n \in \Re$,

$$C(A_1 \cup A_2 \cup \ldots A_n) \geq \sum_i C(A_i) - \sum_{i>j} C(A_i \cap A_j) \ldots . -(-1)^n C(A_1 \cap A_2 \cap \ldots A_n) \qquad \text{(App1.1)}$$

When C is a monotone capacity of order n, it is also a monotone capacity of order m for every $m \leq n$. A monotone capacity of order 1 is defined as a capacity monotone for inclusion (as in requirement A2.1). A monotone capacity of order 2 is called a convex capacity. A belief function is a monotone capacity of order infinite.

Notice that we do not require that $C(\Omega) = 1$ as is usually accepted. We can have $C(\Omega) < 1$. All properties of capacities used in this paper do not depend on $C(\Omega)$ being 1 or less than 1.

The Möbius transform v of a capacity is the function on $\Re$ with:

$$v(A) = \sum_{B: B \in \Re, \emptyset \neq B \subseteq A} (-1)^{|A|-|B|} C(B) \qquad \forall A \in \Re, A \neq \emptyset \qquad \text{(App1.2)}$$

$$v(\emptyset) = 1 - C(\Omega)$$

If C is monotone capacity of order n, then $v(A) \geq 0$ for every A in $\Re$ with $|A| \leq n$.

The functions C and v are in one-to-one correspondence with:

$$C(A) = \sum_{B: B \in \Re, \emptyset \neq B \subseteq A} v(B) \qquad \forall A \in \Re, A \neq \emptyset \qquad \text{(App1.3)}$$

$$C(\emptyset) = 0.$$

Our definitions are based on $C(\emptyset) = 0$, whereas $v(\emptyset)$ might be positive. Adaptation with $C(\emptyset) > 0$ is straightforward, but unused in this paper.

Appendix 2: Proofs of the theorems.

Proof of Theorem 5.

1: Let $(\Omega, \Re, Cr, EC)$ and $(\Omega, \Re, Cr_A, EC \oplus Ev_A)$ be two belief states where $A \in \Re$ and Ev_A is the piece of evidence such that $A =_{Ev_A} \Omega$. As $\overline{A} =_{Ev_A} \emptyset$, requirement A2.1 implies that $Cr_A(\overline{A}) = 0$ where $\alpha_\perp = 0$ (see section 3.3) and for $B \in \Re$, $Cr_A(B) = 0$ $\forall B \subseteq \overline{A}$.

2: After conditioning on A, $A \cap B =_{Ev_A} B$ for all $B \in \Re$. Then by requirement A2.2,

$$Cr_A(B) = Cr_A(A \cap B).$$

3: Let A, B$\in\Re$, B$\subseteq$A. Let a coarsening C from $\Re$ to $\Re'$ such that $\Re'$ has three atoms: C(B), C($\overline{B}\cap A$) and C($\overline{A}$). Consider the belief states (Ω, $\Re'$, Cr', EC) and (Ω, $\Re'$, Cr'_A, EC$\oplus Ev_A$) derived from (Ω, $\Re$, Cr, EC) and (Ω, $\Re$, Cr_A, EC$\oplus Ev_A$) by the coarsening C, respectively. As $\Re'$ has only three atoms, Cr' is completely defined by its values on the seven elements of $\Re'$ (on Ø, Cr(Ø)=0). Hence by requirement A5.1, Cr'_A is defined by these seven values of Cr'. As A $=_{EC}$ C(A), B $=_{EC}$ C(B), …

$$Cr_A(B) = Cr'_{C(A)}(C(B)).$$

The equality results from the fact that $Cr'_{C(A)}$ is a coarsening of Cr_A.
As the RHS depends only on these seven values of Cr', so the LHS depends only on the corresponding elements in $\Re$ as presented in property 3 of theorem 5, except it has been written for any B. But thanks to property 2 of theorem 5 the difference is irrelevant. QED

Lemma 1. Given two belief states (Ω, $\Re'$, Cr', EC) and (Ω, $\Re''$, Cr'', EC) where $|At(\Re')| = |At(\Re'')|$ and the elements X_i and Y_i, i=1,2...n, of $At(\Re')$ and $At(\Re'')$ are so ordered that $X_i =_{EC} Y_i$ for all i=1,2...n. Then for any pairs (X,Y) and (A,B) where X$\in\Re'$, Y$\in\Re''$, A$\in\Re'$, B$\in\Re''$, X $=_{EC}$ Y and A $=_{EC}$ B, we have:

$$Cr'_X(A) = Cr''_Y(B).$$

Proof: Write $Cr'_X(A)$ and $Cr''_Y(B)$ with the function f of theorem 5.3. Given the doxastic equivalence between the elements of $\Re'$ and $\Re''$, corresponding terms in the two f functions share the same numerical value. Hence the two functions are numerically equal. QED

Lemma 2: Let (Ω, $\Re$) be a propositional space and Cr, Cr' and Cr'' be three credibility functions defined on $\Re$ with:

$$Cr = \alpha\, Cr' + (1-\alpha)\, Cr'',$$

for $\alpha\in[0,1]$. Let the conditioning proposition Ev_A, and let Cr_A, Cr'_A and Cr''_A be the conditional credibility functions induced by the conditioning of Cr, Cr' and Cr'' on Ev_A, respectively. The homomorphism requirement A5.3 implies:

$$Cr_A(X) = \sum_{Y\in\Re} c_{A,X}(Y)\, Cr(Y),$$

where the coefficients $c_{A,X}(Y)$ do not depend on Cr.
Proof. By the requirement A5.1, there exists a f_A function such that:

$$Cr_A = f_A(\{Cr(X): X\in\Re\}).$$

The homomorphism requirement becomes:

$$f_A(\{\alpha\, Cr'(X) + (1-\alpha)\, Cr''(X): X\in\Re\}) = \alpha.f_A(\{Cr'(X): X\in\Re\}) + (1-\alpha).f_A(\{Cr''(X): X\in\Re\})$$

where f_A is bounded. Hence f_A is a linear function of its components (see Aczel, 1966, Chapter 5.1., theorem 2 and Chapter 2.1.1, theorem 1) QED

Lemma 3. Given lemma 2 and theorem 5, there exists coefficients γ_i , i=0, 1...7 and α_i, i=0,1,2,3, such that for A$\neq$Ø, B$\subset$A, A,B$\in\Re$,

$$Cr_A(B) = \gamma_0+\gamma_1Cr(B\cap A)+\gamma_2Cr(\overline{B}\cap A)+\gamma_3Cr(\overline{A})+\gamma_4Cr(A)+\gamma_5Cr(B\cup\overline{A})+\gamma_6Cr(\overline{B}\cup\overline{A})+\gamma_7Cr(\Omega),$$

$$Cr_A(A) = \alpha_0+\alpha_1Cr(A)+\alpha_2Cr(\overline{A})+\alpha_3Cr(\Omega).$$

Proof: By theoram 5 and lemma 2, we know that $Cr_A(B)$ and $Cr_A(A)$ depend linearly only on those Cr terms listed in the relations of this lemma. By definition of the conditioning process, $A \neq \emptyset$. Given the invariance of the results to coarsening, the only relevant cases for A and B are:

1: $A=\Omega$, in which case $Cr_\Omega(B) = Cr(B)$, for all $B \in \mathfrak{R}$.

2: $A \neq \Omega$, $A \neq \emptyset$, $B=A$, in which case $Cr_A(A)$ depends on coefficients $c_{A,A}(Y)$ that depends on Y and A, as it is the case with the α_i coefficients.

3: $A \neq \Omega$, $A \neq \emptyset$, $B \subset A$, in which case $Cr_A(B)$ depends on coefficients $c_{A,B}(Y)$ that depends on Y, A and B, as it is the case with the γ_i coefficients.

4: $A \neq \Omega$, $A \neq \emptyset$, $B=\emptyset$, in which case $Cr_A(\emptyset) = 0$.

So there are only two relevant relations, those in the present lemma. QED

Lemma 4: The preservation requirement implies that

X_1: $\gamma_0 + \gamma_1 + \gamma_4 + \gamma_5 + \gamma_7 = 1$

X_2: $\alpha_0 + \alpha_1 + \alpha_3 = 1$

X_3: $\gamma_0 + \gamma_3 + \gamma_5 + \gamma_6 + \gamma_7 = 0$

X_4: $\alpha_0 + \alpha_2 + \alpha_3 = 0$

X_5: $\gamma_2 - \gamma_3 + \gamma_4 - \gamma_5 = 0$.

X_6: $\gamma_0 + \gamma_2 + \gamma_4 + \gamma_6 + \gamma_7 = 0$

Proof: Consider an algebra $\mathfrak{R}$ with four atoms A_1, A_2, A_3, A_4. If $Cr(A_1 \cup A_2) = Cr(A_1 \cup A_2 \cup A_3 \cup A_4)$, the preservation requirement implies that:

$$Cr_{A_1 \cup A_3}(A_1) = Cr_{A_1 \cup A_3}(A_1 \cup A_3). \qquad (*)$$

The results are obtained by explicitating the various terms with lemma 3 results, and studying particular cases.

Suppose $Cr(A_1) = 1$, then $Cr_{A_1 \cup A_3}(A_1) = 1$ implies the equalities X_1 and X_2.

Suppose $Cr(A_2) = 1$, then (*) implies that $\gamma_0 + \gamma_3 + \gamma_5 + \gamma_6 + \gamma_7 = \alpha_0 + \alpha_2 + \alpha_3$.

Suppose $Cr(A_2) = p$ and $Cr(A_3) = 1-p$. Then $Cr_{A_1 \cup A_2}(A_1) = 0$, what implies the relations X_3 and X_4. Relation X_6 results from X_3 and X_5. QED

Lemma 5. In lemma 3, the coefficients of Cr_A satisfy:

(c_0) $\gamma_0 = \gamma_0 + \gamma_0.(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + \gamma_6) + \gamma_7.\alpha_0$

(c_1) $\gamma_1 = \gamma_1.\gamma_1 + \gamma_2.\gamma_6$

(c_2) $\gamma_2 = \gamma_1.\gamma_2 + \gamma_2.\gamma_5$

(c_3) $0 = \gamma_1.\gamma_3 + \gamma_2.\gamma_4$

(c_4) $0 = \gamma_3.(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + \gamma_6) + \gamma_7.\alpha_2$

(c_{12}) $\gamma_4 = \gamma_1.\gamma_4 + \gamma_2.\gamma_3$

(c_{13}) $0 = \gamma_1.\gamma_5 + \gamma_2.\gamma_2$

(c_{14}) $0 = \gamma_1.\gamma_5 + \gamma_6.\gamma_6$

(c_{23}) $0 = \gamma_1.\gamma_6 + \gamma_1.\gamma_2$

(c_{24}) $0 = \gamma_2.\gamma_5 + \gamma_5.\gamma_6$

(c_{34}) $\gamma_3 = \gamma_3.\gamma_5 + \gamma_4.\gamma_6$

(c_{123}) $0 = \gamma_4.(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + \gamma_6) + \gamma_7.\alpha_1$

(c_{124})	$0 = \gamma_4.\gamma_5 + \gamma_3.\gamma_6.$
(c_{134})	$0 = \gamma_2.\gamma_6 + \gamma_5.\gamma_5.$
(c_{234})	$0 = \gamma_1.\gamma_6 + \gamma_5.\gamma_6.$
(c_{1234})	$\gamma_7 = \gamma_7.(\gamma_1 + \gamma_2+ \gamma_3+ \gamma_4+ \gamma_5+ \gamma_6+ \gamma_7) + \gamma_7.\alpha_3$
(a_0)	$\gamma_0 (\alpha_1 + \alpha_2) = - \alpha_0 \alpha_3$
(a_1)	$\alpha_1 = \alpha_1 \gamma_1 + \alpha_2 \gamma_2$
(a_3)	$0 = \alpha_1 \gamma_2 + \alpha_2 \gamma_1$
(a_4)	$0 = (\alpha_1 + \alpha_2) \gamma_3 + \alpha_2 \alpha_3$
(a_{13})	$0 = (\alpha_1 + \alpha_2) \gamma_4 + \alpha_1 \alpha_3$
(a_{14})	$0 = \alpha_1 \gamma_5 + \alpha_2 \gamma_6$
(a_{34})	$\alpha_2 = \alpha_1 \gamma_6 + \alpha_2 \gamma_5$
(a_{134})	$\alpha_3 = (\alpha_1 + \alpha_2) \gamma_7 + \alpha_3 \alpha_3.$

Proof. Let $(\Omega, \Re, Cr)$ be credibility space where A_i: i=1,…4 are the atoms of $\Re$. Let $c_i = Cr(A_i)$, $c_{ij} = Cr(A_i \cup A_j)$, $c_{ijk} = Cr(A_i \cup A_j \cup A_k)$, $c_{1234} = Cr(A_1 \cup A_2 \cup A_3 \cup A_4)$ where $i \neq j \neq k \in \{1, 2, 3, 4\}$

By conditioning on $A_1 \cup A_2$ or successively on $A_1 \cup A_2 \cup A_3$ and the result on $A_1 \cup A_2$, we get the same result by theorem 4. Using the function f of theorem 5 (3) and using the notation Cr(B|A) for $Cr_A(B)$, we get:

$Cr(A_1|A_1 \cup A_2) = f(c_1, c_2, c_{34}, c_{12}, c_{134}, c_{234}, c_{1234}) = f(a_1, a_2, a_3, a_{12}, a_{13}, a_{23}, a_{123})$

with $a_1 = Cr(A_1|A_1 \cup A_2 \cup A_3) = f(c_1, c_{23}, c_4, c_{123}, c_{14}, c_{234}, c_{1234})$

$a_2 = Cr(A_2|A_1 \cup A_2 \cup A_3) = f(c_2, c_{13}, c_4, c_{123}, c_{24}, c_{134}, c_{1234})$

$a_3 = Cr(A_3|A_1 \cup A_2 \cup A_3) = f(c_3, c_{12}, c_4, c_{123}, c_{34}, c_{124}, c_{1234})$

$a_{12} = Cr(A_1 \cup A_2|A_1 \cup A_2 \cup A_3) = f(c_{12}, c_3, c_4, c_{123}, c_{124}, c_{34}, c_{1234})$

$a_{13} = Cr(A_1 \cup A_3|A_1 \cup A_2 \cup A_3) = f(c_{13}, c_2, c_4, c_{123}, c_{134}, c_{24}, c_{1234})$

$a_{23} = Cr(A_2 \cup A_3|A_1 \cup A_2 \cup A_3) = f(c_{23}, c_1, c_4, c_{123}, c_{234}, c_{14}, c_{1234})$

$a_{123} = Cr(A_1 \cup A_2 \cup A_3|A_1 \cup A_2 \cup A_3) = f(c_{123}, 0, c_4, c_{123}, c_{1234}, c_4, c_{1234})$

Rewriting each term with the relation of lemma 3, and collecting the coefficients of the c terms indicated in the label of each equality of the lemma, one obtains the requested equalities.
Repeating the construction after eliminating A_2 form Ω and considering $Cr(A_1|A_1)$, one gets the (a_i) equalities. QED

Lemma 6. Given lemmas 3 and 4, there are only three solutions for Cr_A.

1) $Cr_A(B) = Cr(A \cap B)$

2) $Cr_A(B) = Cr(B \cup \overline{A}) - Cr(\overline{A})$

3) $Cr_A(B) = \beta.Cr(B \cap A) + (1-\beta).(Cr(B \cup \overline{A})-Cr(\overline{A})) + \alpha.(Cr(\Omega)-Cr(A)-Cr(\overline{B} \cup \overline{A})+Cr(\overline{B} \cap A))$, where $-\alpha^2 = \beta.(1-\beta)$.

Proof: In this proof, X_i denotes the lemma 4 equalities, and (c_i) denotes the lemma 5 equalities. Relations (c_{13}) and (c_{14}) of lemma 5 imply that $\gamma_2 = \gamma_6$ or $\gamma_2 = - \gamma_6$.

Case 1: Suppose $\gamma_2 = \gamma_6 = 0$. By (c_1), $\gamma_1 = 0$ or $\gamma_1 = 1$.

Case 1.1: let $\gamma_1 = 1$. By (c_{13}), $\gamma_5 = 0$. By (c_3), $\gamma_3 = 0$. By X_5, $\gamma_4 = 0$. By (c_0), (c_4), (c_{123}) and (c_{1234}), we obtain: $\gamma_0 + \gamma_7.\alpha_0 = 0$, $\gamma_7.\alpha_2 = 0$, $\gamma_7.\alpha_1 = 0$, $\gamma_7.\alpha_3 = 0$. If $\gamma_7 \neq 0$, $\alpha_1 = \alpha_2 = \alpha_3 = 0$. then by X_2, $\alpha_0 = 1$ what contradicts X_4. Hence $\gamma_7 = 0$. By X_6, $\gamma_0 = 0$.
Hence $Cr_A(B) = Cr(A\cap B)$ if $B\subset A$.
By (a_3), $\alpha_2 = 0$. By (a_{134}), $\alpha_3 = 1$ or 0.
Case 1.1.a: if $\alpha_3 = 1$, by X_4, $\alpha_0 = -1$, what contradicts (a_0) where $\alpha_0\alpha_3 = 0$.
Case 1.1..b: If $\alpha_3 = 0$, by X_4, $\alpha_0 = 0$, and by X_2, $\alpha_1 = 1$.
Hence $Cr_A(A) = Cr(A)$.
Case 1.2: let $\gamma_1 = 0$. By (c_{12}), $\gamma_4 = 0$. By (c_{134}), $\gamma_5 = 0$ or $\gamma_5 = 1$.
Case 1.2.a: let $\gamma_5 = 0$. By (c_{34}), $\gamma_3 = 0$. By X_1, $\gamma_0 + \gamma_7 = 1$, in contradicion with X_3.
Case 1.2.b: let $\gamma_5 = 1$. By X_1, $\gamma_0 + \gamma_7 = 1$. By X_3, $\gamma_3 = -1$. By (a_1), $\alpha_1 = 0$. By X_2, $\alpha_0 + \alpha_3 = 1$. By X_4, $\alpha_2 = -1$. By (c_4), $\gamma_7 = 0$. Then $\gamma_0 = 0$. By (a_4), $\alpha_3 = 1$. Then, $\alpha_0 = 0$. Hence $Cr_A(B) = Cr(B\cup\overline{A}) - Cr(\overline{A})$, for all $B\subseteq A$, $B\in\mathfrak{R}$.
Case 2: Let $\gamma_2 = \gamma_6 = \alpha \neq 0$. By ($c_{23}$), $\gamma_1 = 0$. By (c_1), $0 = \alpha^2$, what contradicts the initial assumption.
Case 3: Let $\gamma_2 = -\gamma_6 = \alpha \neq 0$. Let $\gamma_1 = \beta$. Then (c_1) gives: $\beta(1-\beta) = -\alpha^2$, and $\alpha\neq0$ implies that $\beta\notin[0,1]$. By X_6, $\gamma_0 + \gamma_5 + \gamma_7 = 0$. Then by X_1, $\gamma_1 + \gamma_5 = 1$, hence $\gamma_5 = 1-\beta$.
Case 3.1: Let $\gamma_7 = 0$. By (a_{134}), $\alpha_3 = 0$ or $\alpha_3 = 1$.
Case 3.1.1: Let $\alpha_3 = 0$.
Case 3.1.1.a: let $\alpha_1 + \alpha_2 \neq 0$. By ($a_0$), ($a_4$) and ($a_{14}$), $\gamma_0 = \gamma_3 = \gamma_4 = 0$. By X_3, $(1-\beta) - \alpha = 0$. As $\beta(1-\beta) = -\alpha^2$, we get $-\beta(1-\beta) = (1-\beta)^2$, thus $\beta = 1$, what is not allowed as $\beta\notin[0,1]$.
Case 3.1.1.b: let $\alpha_1 + \alpha_2 = 0$. By X_2 and X_4, $\alpha_0 = \alpha_1 = -\alpha_2 = .5$. By ($a_3$), $\alpha = \beta$. Together with $\beta(1-\beta) = -\alpha^2$, it means $\beta = 0$, what is not allowed as $\beta\notin[0,1]$.
Case 3.1.2: Let $\alpha_3 = 1$.
Case 3.1.2.a: Let $1 + \gamma_3 + \gamma_4 \neq 0$. By ($c_0$), ($c_4$) and ($c_{123}$), $\gamma_0 = \gamma_3 = \gamma_4 = 0$. By X_3, $(1-\beta) - \alpha = 0$, what has been shoxn to be not acceptable in case 3.1.1.a.
Case 3.1.2.b: Let $1 + \gamma_3 + \gamma_4 = 0$. Let $\delta = \gamma_4$, then $\gamma_3 = -1-\delta$, and $\gamma_0 = -\delta$ by X_1. By X_3, $2\delta + \alpha + \beta = 0$. By ($c_{12}$) and ($c_{34}$), $\delta(1-\beta) + \alpha(1+\delta) = 0$ and $(1+\delta)\beta = \alpha\delta$. Hence, $\delta+\alpha+\beta = 0$. So $2\delta = \delta$, hence $\delta = 0$. By (c_{12}), $\alpha = 0$, contrary to the initial assumption.
Case 3.2: Let $\gamma_7 \neq 0$. Let $x = \gamma_3$. Then by X_5, $\gamma_4 = x + 1 - \alpha - \beta$. By ($c_3$), $\gamma_4 = -\beta x/\alpha$. Hence, $x = (\alpha+\beta-1)\alpha/(\alpha+\beta)$. Given $\beta(1-\beta) = -\alpha^2$, it reduces to $x = \beta-1$. Hence $\gamma_3 = \beta-1$ and $\gamma_4 = -\alpha$. By (c_{1234}), $\alpha_3 = 1+\alpha-\beta$. The difference of the products of $1-\beta$ by X_2 and of α by X_4 gives: $\alpha_0 = 0$. By (c_0), $\gamma_0 = 0$. By X_4, $\alpha_2 = \beta-1-\alpha$. By X_2, $\alpha_1 = \beta-\alpha$. By X_1, $\gamma_7 = -\alpha$. In such a case:
$Cr_A(B) = \beta.Cr(B\cap A) - (1-\beta).(Cr(B\cup\overline{A})-Cr(\overline{A})) + \alpha.(Cr(\Omega)-Cr(A)-Cr(\overline{B}\cup\overline{A})+Cr(\overline{B}\cap A))$.

QED

CHANGER NUMERO XXXX

Theorem 6: Assume the conditions and notations of requirement A5.2. Let Cr' be the credibility function derived from Cr on $\mathfrak{R}'$ by R. Then for all $D,X\in\mathfrak{R}$:

$$Cr'_{R(D)\cap\overline{\mathbf{B}}}(R(X)\cap\overline{B}) = Cr_D(X).$$

Proof: R(D) and D, R(X) and X are doxastically equivalent by construction. Hence $Cr'_{R(D)}(R(X)) = Cr_D(X)$. $R(X)\cap\overline{B}$ and R(X) are doxastically equivalent under EC_1 by requirement A5.2, hence $Cr'(R(X)\cap\overline{B}) = Cr'(R(X))$. This equality persists after conditioning both terms on R(D), hence $Cr'_{R(D)}(R(X)\cap\overline{B}) = Cr'_{R(D)}(R(X))$. R(D) and $R(D)\cap\overline{B}$ are also doxastically equivalent under EC_1 by requirement A5.2, hence $Cr'_{R(D)\cap\overline{B}}(R(X)\cap\overline{B}) = Cr'_{R(D)}(R(X)\cap\overline{B})$. Combining these equalities proves the theorem. QED

Lemma 7. Let $(\Omega, \Re, Cr)$ be credibility space where $\Re$ has two atoms A_1, A_2. Let R be a uninformative refinement from $\Re$ to $\Re'$ such that $\Re$ has three atoms B_1, B_2 and B_3 with $R(A_1) = B_1$ and $R(A_2) = B_2\cup B_3$. Let Cr' be the credibility function induced from Cr by R on $\Re'$. Then according to the three solutions of lemma 6, Cr' satisfies:

1) $Cr'(B_2) = Cr(A_2)$ and $Cr'(B_1\cup B_2) = Cr(A_1\cup A_2)$
2) $Cr'(B_2) = 0$ and $Cr'(B_1\cup B_2) = Cr(A_1)$
3) no solution.

Proof: By theorem 6, with $D=\Omega$ and B being successively $B_1\cup B_2$ and $B_1\cup B_3$, we have

$Cr(A_1) = Cr'(B_1|B_1\cup B_2) = Cr'(B_1|B_1\cup B_3)$ *
$Cr(A_2) = Cr'(B_2|B_1\cup B_2) = Cr'(B_3|B_1\cup B_3)$ **
$Cr(A_1\cup A_2) = Cr'(B_1\cup B_2|B_1\cup B_2) = Cr'(B_1\cup B_3|B_1\cup B_3)$ ***

We consider successively the three solutions of lemma 6.

Solution 1: $Cr_A(B)= Cr(A\cap B)$.
By **, $Cr(A_2) = Cr'(B_2) = Cr'(B_3)$.
By ***, $Cr(A_1\cup A_2) = Cr'(B_1\cup B_2) = Cr'(B_1\cup B_3)$.

Solution 2 : $Cr_A(B) = Cr(B\cup\overline{A}) - Cr(\overline{A})$
By **, $Cr(A_2) = Cr'(B_2\cup B_3) - Cr'(B_3) = Cr'(B_2\cup B_3) - Cr'(B_2)$,
hence $Cr'(B_2) = Cr'(B_3)$. As $Cr'(B_2\cup B_3) = Cr(A_2)$, then $Cr'(B_2) = 0$.
By *, $Cr(A_1) = Cr'(B_1\cup B_3) - Cr'(B_3) = Cr'(B_1\cup B_2) - Cr'(B_2)$,
hence $Cr(A_1) = Cr'(B_1\cup B_3) = Cr'(B_1\cup B_2)$.

Solution 3 : $Cr_A(B) = \beta.Cr(B\cap A) + (1-\beta).(Cr(B\cup\overline{A})-Cr(\overline{A}))$
$+ \alpha.(Cr(\Omega)-Cr(A)-Cr(\overline{B}\cup\overline{A})+Cr(\overline{B}\cap A))$.
Replacing the different terms in ** by their solutions, we get $Cr'(B_2) = Cr'(B_3)$.
Doing the same with * and using the last equality leads to $(\alpha+1-\beta).(Cr'(B_1\cup B_2)-Cr'(B_1\cup B_3)) = 0$. As $\alpha+1-\beta = 0$ is not compatible with $\alpha^2 = -\beta+\beta^2$, (see the assumptions that led to solution 3 in lemma 6), $Cr'(B_1\cup B_2) = Cr'(B_1\cup B_3)$.
Introducing these constraints in *, ** and ***, one gets respectively:
$0 = (1-\beta).(Cr'(B_1\cup B_2)-Cr'(B_2)-Cr(A_1)) + \alpha.(Cr(A_1\cup A_2) - Cr'(B_1\cup B_2) - Cr(A_2) + Cr'(B_2))$
$0 = -(1-\beta).Cr'(B_2) - \beta.(Cr(A_2) - Cr'(B_2)) + \alpha.(Cr(A_1\cup A_2) - 2.Cr'(B_1\cup B_2) + Cr(A_1))$
$0 = \beta.Cr'(B_1\cup B_2) - Cr'(B_2) -\beta.(Cr(A_1\cup A_2) - Cr'(B_2)) +$

$\alpha(Cr(A_1 \cup A_2) - Cr'(B_1 \cup B_2) - Cr'(B_2))$

The solution of these three equations are:

$Cr'(B_2) = \alpha.(Cr(A_1 \cup A_2)-Cr(A_1)) + \beta.Cr(A_2)$

$Cr'(B_1 \cup B_2) = (1-\beta).Cr(A_1) + \alpha.Cr(A_2) + \beta.Cr(A_1 \cup A_2)$

Let R' be a uninformative refinement from $\mathfrak{R}'$ to $\mathfrak{R}''$ such that $\mathfrak{R}''$ has four atoms C_1, C_2, C_3 and C_4 with $R'(B_1) = C_1$ and $R'(B_2) = C_2 \cup C_3$, and $R'(B_3) = C_4$. Let Cr'' be the credibility function induced from Cr' by R' on $\mathfrak{R}''$. Applying the previous procedure with a conditioning on $C_1 \cup C_2$ leads to:

$$Cr''(C_2) = \alpha.(Cr'(B_1 \cup B_2)-Cr'(B_1)) + \beta.Cr'(B_2)$$
$$= \alpha.(Cr(A_1 \cup A_2) - Cr(A_1)) + \beta.Cr(A_2)$$

and

$$Cr''(C_1 \cup C_2) = (1-\beta).Cr'(B_1) + \alpha.Cr'(B_2) + \beta.Cr'(B_1 \cup B_2)$$
$$= (1-\beta).Cr(A_1) + \alpha.Cr(A_2) + \beta.Cr(A_1 \cup A_2)$$

where the second equalities in both relation results from considering the direct refinement from $\mathfrak{R}$ to $\mathfrak{R}''$.

Collecting the terms, one gets:

$0 = (2\alpha\beta-\alpha).(Cr(A_1 \cup A_2)-Cr(A_1))+ 2\alpha^2.Cr(A_2)$

$0 = 2\alpha^2.(Cr(A_1 \cup A_2)-Cr(A_1)) + (2\alpha\beta-\alpha).Cr(A_2).$

If $Cr(A_1 \cup A_2)-Cr(A_1) = Cr(A_2)$ (like with probability functions), one gets:
$2\alpha\beta - \alpha + 2\alpha^2 = 0$, i.e., either $\alpha=0$ or $\alpha+\beta=1/2$,
otherwise, one gets:
$0 = (2\alpha^2-2\alpha\beta+\alpha).(Cr(A_1 \cup A_2)-Cr(A_1)-Cr(A_2))$, i.e., $\alpha=0$ or $\alpha-\beta=-1/2$
In both cases, the solutions for α and β are incompatible with $\alpha \neq 0$ and $\alpha^2=-\beta+\beta^2$ that underlies the third solution of theorem 5. Hence the third solution does not admit a solution.

QED

Lemma 8. In lemma 7, solution 1 does not satisfy the preservation requirement, solution 2 does it.

Proof:

Solution 1: Let $(\Omega, \mathfrak{R}, Cr)$ be a credibility space where $\mathfrak{R}$ has two atoms: A_1 and A_2, with $Cr(A_1) = 0$ and $Cr(A_2) = 1$. This Cr function is indeed a credibility function as it corresponds to the probability function with probability 1 given to A_2, and probability functions are credibility functions.
Let R be an uninformative refinement from $\mathfrak{R}$ to $\mathfrak{R}'$ where $\mathfrak{R}'$ has three atoms B_1, B_2 and B_3 such that $R(A_1) = B_1$, $R(A_2) = B_2 \cup B_3$, and let Cr' be the credibility function induced from Cr by R on $\mathfrak{R}'$. Then $Cr'(B_1) = Cr(A_1) = 0$, $Cr'(B_2 \cup B_3) = Cr(A_2) = 1$.
Under solution 1 and with the results of lemma 7, $Cr'(B_2) = Cr'(B_3) = Cr'(B_2 \cup B_3) = 1$.
Hence $Cr'(B_1 \cup B_2) = Cr'(B_1 \cup B_2 \cup B_3) = Cr'(\Omega)$.
By lemma 7, solution 1,

$Cr_{B_1 \cup B_3}(B_1 \cup B_2) = Cr'(B_1) = 0,$

$Cr_{B_1 \cup B_3}(B_1 \cup B_3) = Cr'(B_1 \cup B_3) = 1$,

Preservation would have required that $Cr_{B_1 \cup B_3}(B_1 \cup B_2) = Cr_{B_1 \cup B_3}(B_1 \cup B_3)$, what is not satisfied. Hence, solution 1 violates the preservation requirement.

Solution 2: By lemma 5, solution (2) gives:

$Cr_A(B) = Cr(B \cup \overline{A}) - Cr(\overline{A})$

$Cr_A(A) = Cr(\Omega) - Cr(\overline{A})$.

By the monotony requirement in A2.1,

$Cr(\Omega) \geq Cr(B \cup \overline{A}) \geq Cr(B)$.

If $Cr(\Omega) = Cr(B)$, then $Cr(B \cup \overline{A}) = Cr(\Omega)$ in which case $Cr_A(B) = Cr_A(A)$. QED

Proof of theorem 7: Immediate as it presents the solutions of lemma 6 and 7 that satisfy lemma 8. QED

Comments: Solution 1 (2) corresponds to the formula encountered in Dempster-Shafer models and the TBM if Cr is a plausibility (belief) function. The case of the plausibility function is rejected as it violates the preservation solution. The only solution left over is the belief function solution. The next lemma generalized the previous results and is obtained by iterating both the conditioning and the refinement processes.

Proof of theorem 8. Let $X, Y \in \Re$. Let $\overline{A} = X \cap Y$, $C = \overline{X} \cap Y$, $\overline{B} = X \cap \overline{Y}$.

By construction, $A, B, C \in \Re$. Then

$Cr_{A \cap B}(C) = Cr(C \cup \overline{B} \cup \overline{A}) - Cr(\overline{B} \cup \overline{A}) \geq Cr_A(C) = Cr(C \cup \overline{A}) - Cr(\overline{A})$

Thus: $Cr(X \cup Y) \geq Cr(X) + Cr(Y) - Cr(X \cap Y)$,

which proves that Cr is a convex capacity. QED

Proof of Theorem 9: Take the highly degenerated propositional space $(\Omega, \Re)$ such that $\Re$ contains only one atom, Ω. As far as probability functions are credibility functions, the credibility function with:

$Cr(\Omega) = 1, \qquad Cr(\emptyset) = 0$

belongs to the set of credibility functions over this highly degenerated space.

Through uninformative refinements and conditionings, one can generate for any $A \in \Re$, the credibility function Cr_A such that:

$$Cr_A(X) = 1 \quad \text{if } A \subseteq X, X \in \Re$$
$$= 0 \quad \text{otherwise}$$

Let bel be a belief function defined on $\Re$ and let v be its Möbius transform. By definition of a belief function (appendix 1), $v(X) \geq 0$ for every $X \in \Re$ and $\sum_{X \in \Re} v(X) = 1$. Build the function Q on $\Re$ such that:

$$Q(X) = \sum_{A \in \Re} v(A)\, Cr_A(A).$$

By construction $Q(X) = bel(X)$ for every $X \in \Re$.

By theorem 2, Q is a credibility function. Hence bel is a credibility function on $\Re$. □

Proof of theorem 10: The proof is just a formal repetition of the arguments presented in appendix 3.

Appendix 3. The deconditionalization.

Let $(\Omega, \Re)$ be a propositional space and let $N = 2^n$ where n is the number of atoms in $\Re$. Let *Cr* be the set of credibility functions defined on $\Re$. For $X \in \Re$, let Cr_X be the set of conditional credibility functions obtained by conditioning the elements of *Cr* on Ev_X (see section 5). Given $Cr \in$ *Cr* and $Cr_X \in Cr_X$, let v and v_X be their Möbius transforms, respectively. Relation (5.4) gives:

$$v_X(A) = \sum_{B:B \subseteq \overline{X}} v(A \cup B), \qquad \text{(App3.1)}$$

a relation proved in Shafer (1976) (with v being Your 'basic probability assignment').

Let $\mathbf{v}$ and $\mathbf{v_X}$ be the column vectors which elements are the values of v and v_X, respectively. We call them Möbius vectors. For simplicity sake, we write $\mathbf{v} \in$ *Cr* ($\mathbf{v_X} \in Cr_X$) to mean that the vector corresponds to a credibility function that belong to *Cr* (Cr_X).

Relation (App3.1) expressed for $\mathbf{v_X}$ can be written under matricial notation. Let $\mathbf{S_X}$ be the NxN matrix which elements $s: \Re x \Re \rightarrow [0,1]$ are:

$s(A,B) = 1$ if $A \subseteq X$, and there is a $C \subseteq \overline{X}$ such that $B = A \cup C$.

0 otherwise.

Then (App3.1) becomes:

$$\mathbf{v_X} = \mathbf{S_X}.\mathbf{v} \qquad \text{(App3.2)}$$

and (6.2) becomes:

$$\mathbf{S_X}.\mathbf{S_Z} = \mathbf{S_{X \cap Z}}$$

Let $\mathfrak{S}_{\bar{X}}$ be the set of NxN matrices $\mathbf{S_{\bar{X}}}$ which satisfy the constraints expressed by (6.3) and (6.4), i.e.,

$$\mathbf{S_X}.\mathbf{S_{\bar{X}}}.\mathbf{S_X} = \mathbf{S_X} \qquad \text{(App3.3)}$$

$$\mathbf{S_{\bar{X}}}.\mathbf{S_{\bar{X}}} = \mathbf{S_{\bar{X}}} \qquad \text{(App3.4)}$$

We first show that if one element of $\mathbf{v_X}$ is negative, then there is a matrix $\mathbf{S_{\bar{X}}} \in \mathfrak{S}_{\bar{X}}$ such that $\mathbf{S_{\bar{X}}}.\mathbf{v_X}$ is not the Möbius transform of a credibility function. Suppose $v_X(A) < 0$ for $A \in \Re$. Let $\mathbf{S_{\bar{X}}*}$, with elements $s^{-}*(B,C)$, be such that:

$s^{-}*(A,A) = 1$

$s^{-}*(B \cup \overline{X},B) = 1$ for all $B \subseteq X$, $B \neq A$, $B \in \Re$

$s^{-}*(C,C) = 1$ for all $C \not\subseteq X$

$s^{-}*(B,C) = 0$ otherwise.

By construction, $\mathbf{S_{\bar{X}}*}$ satisfies App3.3 and App3.4, hence $\mathbf{S_{\bar{X}}*} \in \mathfrak{S}_{\bar{X}}$. The impact of $\mathbf{S_{\bar{X}}*}$ is such that all v_X are transferred to sets not contained in A, except $v_X(A)$ that is left allocated to A. So the value given to A by the credibility function build from the Möbius vector $\mathbf{S_{\bar{X}}*.v_X}$

is negative, thus violating the credibility functions properties obtained in section 3.3. Hence none of the elements of $\mathbf{v_X}$ may be negative. Therefore the elements of Cr_X must be belief functions. As any set of credibility functions *Cr* results itself from a conditioning on an evidential corpus EC, what has been proved for Cr_X can be extended directly to any *Cr*, therefore every *Cr* is a set of belief functions.

We can also prove that the elements of $\mathbf{S^-_X}$ are non negative. For A,B∈ℜ, let $s^-(A,B)$ be a component of $\mathbf{S^-_X}$ with $s^-(A,B) < 0$. Then takes $\mathbf{v_X}$ such that $v_X(A) = 1$, and $v_X(B) = 0$ for all B≠A. Such $\mathbf{v_X}$ is induced by a credibility function as shown in theorem 9. Let $\mathbf{v} = \mathbf{S^-_X}.\mathbf{v_X}$. By construction, the elements of $\mathbf{v}$ are:

$$v(C) = s^-(A,C) \qquad \text{for all } C\in\Re$$

So v(B) < 0. Hence, the deconditioning of $\mathbf{v}$ could lead to a result that is not a credibility function. In order to avoid it, the elements of $\mathbf{S^-_X}$ must be non negative.

It remains to show that every $\mathbf{S^-_X} \in \mathfrak{S}^-_X$ maps any belief function into a belief function. This is immediate as the elements of $\mathbf{S^-_X}$ are non negative.

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