4.6 La thèse d'Artemov

Artemov montre qu'on peut faire une thèse de ce que j'ai appellé le stratagème du Théétète.

Extrait de Artemov 1990.

"Are there any reasons for adopting the definition $\boxdot P := P \& \Box P$? The modality $\boxdot$ doesn't have an explicit mathematical model; it had been introduced as a modality for an intuitive notion of mathematical provability. On the contrary the modality $\Box$ has an exact mathematical definition as an operator of formal provability Pr(.) on the set of arithmetical sentences. Thus there is no way to prove that $\boxdot P := P \& \Box P$; one can only hope to find some arguments in order to declare a

Thesis : $\boxdot P := P \& \Box P$ (* *)

(like the Church Thesis for computable functions). Gödel himself in [Gödel 1933] tried the obvious idea to define $\boxdot Q$ as $\Box Q$ but noticed that this definition led to a contradiction between his axioms and rules for $\boxdot$ and the already known Gödel Second Incompleteness Theorem. Can one nevertheless give a reasonable definition of $\boxdot$ via $\Box$? The most optimistic expectations are

> to find a $\Box$-formula B(p) which satisfies known properties of $\boxdot p$ (first of all axioms and rules of S4) and such that for each other $\Box$-formula C(p) with these properties

$$G \vdash B(p) \leftrightarrow C(p)$$

In this case we have the right to declare a definition $\boxdot Q := B(p)$ as a Thesis. It turns out that this situation holds with $p \& \Box p$ as B(p). The main ideas of the proof of the following theorem were taken from [Kuznetsov & Muravitsky 1986].

Theorem 6. For a given $\Box$-formula C(p) if

1. all axioms and rules of S4 for C(p) as $\boxdot p$ are arithmetically valid (derivable in G) and*

2. $G \vdash C(p) \rightarrow \Box p$ (this principle says that any "real" mathematical proof can be finitely transformed into a formal proof)

then

$$G \vdash C(p) \leftrightarrow (p \& \Box p)$$

Proof. Let $\top$ denotes the propositional constant "truth" so $\top \in$ Int, S4, Grz, G, G*. Obviously, $S4 \vdash \boxdot \top$ and by the conditions of Theorem 6

1) $G^* \vdash C(\top)$,
2) $G^* \vdash C(C(p) \rightarrow p)$ (because $S4 \vdash \boxdot(\boxdot p \rightarrow p)$),
3) for each $\Box$-formula F that contains modality symbols only in combinations of a type C(.)

$$G^* \vdash F \Rightarrow G^* \vdash C(F),$$

(because of the necessitation rule for S4: $S4 \vdash Q \Rightarrow S4 \vdash \boxdot Q$),

4) $G \vdash C(p) \rightarrow \Box p$ (condition 2. of the theorem).

We will show that

$$G \vdash C(p) \leftrightarrow (p \,\&\, \Box p)$$

and thus this formula is deducible in all logics of formal provability. According to 2)

$$G^* \vdash C(C(p) \rightarrow p),$$

thus ($G \subseteq G^*$, condition 2. of the theorem)

$$G \vdash \Box(C(p) \rightarrow p)$$

and

$$G \vdash C(p) \rightarrow p.$$

Together with 4) this gives

$$G \vdash C(p) \rightarrow p \,\&\, \Box p.$$

Lemma. For each $\Box$-formula D(p)

$$G \vdash (p \,\&\, \Box p) \rightarrow (D(p) \leftrightarrow D(\top)).$$

The proof is an induction on the complexity of D. The basis step and induction steps for Boolean connectives are trivial.
Let D(p) be $\Box E(p)$. By the induction hypothesis

$$G \vdash (p \,\&\, \Box p) \rightarrow (E(p) \leftrightarrow E(\top)).$$

The necessitation rule for G and the commutativity of $\Box$ with $\rightarrow$ and & give

$$G \vdash (\Box p \,\&\, \Box \Box p) \rightarrow (\Box E(p) \leftrightarrow \Box E(\top)).$$

Together with $G \vdash \Box p \rightarrow \Box \Box p$ this implies

$$G \vdash (p \,\&\, \Box p) \rightarrow (D(p) \leftrightarrow D(\top)).$$

By 2) $G^* \vdash C(\top)$ and according to 3), 4), $G^* \vdash C(C(\top))$, $G^* \vdash \Box C(p)$ and $G \vdash C(p)$. Because of the lemma we have

$$G \vdash (p \,\&\, \Box p) \rightarrow C(p), \text{ whence } G \vdash C(p) \leftrightarrow (p \,\&\, \Box p).$$

Remark. Without condition 2. of the theorem we lose the uniqueness of the definition (**): C(p):=p also fits."