Université Libre de Bruxelles

*Institut de Recherches Interdisciplinaires
et de Développements en Intelligence Artificielle*

Integrals of Markov Processes with Application to Swarm Robotics Modelling

Y. Khaluf, M. Pace, F. Rammig, and M. Dorigo

IRIDIA – Technical Report Series

Technical Report No.
TR/IRIDIA/2012-020

December 2012

IRIDIA – Technical Report Series
ISSN 1781-3794

Published by:
IRIDIA, *Institut de Recherches Interdisciplinaires et de Développements en Intelligence Artificielle*
Université Libre de Bruxelles
Av F. D. Roosevelt 50, CP 194/6
1050 Bruxelles, Belgium

Technical report number TR/IRIDIA/2012-020

Integrals of Markov processes with application to Swarm Robotics modelling

Yara Khaluf, Michele Pace, Franz Rammig, Marco Dorigo

December 18, 2012

Abstract

This paper reviews some of the techniques for the evaluation of the expected value of a path integral for general Markov chains on countable state spaces and illustrates their application in the context of the analysis of swarm robotics systems. When salient aspects of the dynamics of a multi-robot system can be modelled as a birth-death process, well established results can be used to calculate the expected value as well as the distribution of important quantities such as energy consumption or activity time, even in presence of spatial interferences. By using these results we show that robot controllers can be designed on the basis of birth-death processes and that basic performance guarantees can be derived in closed form.

1 Introduction

Integrals of non-negative stochastic processes arise frequently in engineering applications. Common examples include the estimation of the expected energy consumption by a system subject to random dynamics or the estimation of the cost associated to the execution of a task whose sub-tasks have random durations or costs. Other common applications are found in queuing, storage, and inventory systems [29].
In biology, integrals of non-negative stochastic processes are often associated with the expected resource consumption of groups of animals [25], with the expected amount of toxins produced by bacteria [39], or with the total cost of epidemics [18, 11].
Results in the context of potential theory allows for the characterization of the expected value of this kind of integrals and, in certain cases, for the characterization of the Laplace transform of their distribution. This, in turn, not only mitigates the need of costly numerical simulations, but also allows for the analytical study of important properties of the underlying stochastic processes, especially in relation to complex potential functions.
We investigate the application of these results in the context of swarm robotics.
Swarm robotics is a recent field of research which aims at building robotics systems that are potentially able to solve difficult problems by exploiting a large number of simple robots operating without a central unit of control. By taking inspiration from natural systems such as colonies of social insects or groups of animals, researchers investigate approaches aimed at tackling problems in the context of exploration, search and rescue and structure building, among many others. The central idea is that, like in Nature, scalable and robust solutions can *emerge* based on the simple interactions in a large group of individuals (robots). Moreover, these systems are generally characterized by a degree of robustness and reliability that is difficult to achieve in their single-robot counterparts.
Ants and bees offer the classical example of natural systems in which the interaction of thousands or millions of individuals characterized by relatively simple behaviours is able to generate colonies with the properties of robustness, scalability and adaptability required to colonize an entire planet. When it comes to studying and developing swarm robotics systems, researchers

aim not only at identifying these fundamental behaviours, but also at understanding how they can be used to solve practical problems in different domains. In order to do so, they use their experience and intuition to devise new algorithms or to identify known behaviors suitable for the problem at hand. This approach has lead to the development of algorithms that are able to solve complex optimization and routing problems in an elegant and effective way [7]. However, when it comes to building distributed robotics systems that have to met certain performance guarantees, starting from the design of individual controllers and then checking if the constraints are satisfied may be a tedious and time-consuming process.

In the paper, we consider the opposite approach. Instead of searching for the microscopic behaviour (i.e., the behaviour of the individuals) and for a combination of parameters that allows the system to respect certain constraints, individual robotics controllers are designed to mimic stochastic processes with known analytical properties. This allows the designers to focus on the task of building robotics controllers whose dynamics is as close as possible to the stochastic processes with the desired properties. We investigate the approach in relation to performance characteristics such as power consumption, total activity time, and amount of work obtained by a multi-robot system in presence of spatial interferences.

Although several results discussed in the paper are applicable to general, finite-state, Markov processes, we focus for the most part on birth-death type processes (BDPs).

Given their rich history in modelling natural phenomena, these models have found applications not only in ecology, genetics and biology but also in engineering and physics. In swarm robotics, BDPs have been used to determine the expected time for the robots to cluster [15], to analyse the stability of robotics controllers for the distributed tracking problem [41], and for the modelling of distributed robot deployment as in [27], where the process of gaining and losing robots at different sites follows a birth-death dynamics.

Indirect connections with BDPs exist in works on opinion dynamics for collective decision making [6, 40]. In these works, however, probabilistic considerations are derived via ordinary differential equations (ODE) models. To the best of our knowledge, integrals of birth-death processes have not been investigated in the swarm robotics literature.

The presence of these integrals emerges especially in applications where an important characteristic of the system depends on the number of robots in a certain state, and when the rates at which this number varies are known or measurable. In this case, probabilistic characterizations of these properties can be obtained and used to guide the design towards systems with associated performance guarantees. One example is the total power consumption of a swarm, which, in general, can be related to the number of active robots at each time step. Similarly, when the activation and deactivation of individual robots can be modelled as a birth-death process, the expected duration of an activity can be estimated.

A third example is related to the concept of cost-benefit associated to swarms of different size. The competition for resources (physical space, communication channels, access to power sources, etc.) generates well known interference effects that may reduce the total efficiency of the swarm. The phenomenon may become critical in applications where a large number of robots have to access to the same resources. At the same time, however, and up to a certain point, the increase in the size of the swarm has a positive effect on the performance since more robots are put to work. The non-linearities and the complex dynamics resulting from concurrent effects of cooperation and interference are particularly difficult to study and have been investigated when the size of the swarm does not change in time. In dynamic scenarios where the size of the swarm changes in time, the assessment of the total costs and benefits of a solution can be related to the problem of estimating the integral of a function of a stochastic process.

The rest of the paper is organized as follows. Section 2 reviews relevant publications in the context of the mathematical modelling of swarm robotics systems and provides an overview of the literature of integrals of non-negative stochastic processes. Section 3 introduces a reference model that will be used to illustrate theoretical results, in section 4 the notation used in the paper and some basic results related to birth-death processes. Section 5 focuses on the generalization of these results to the case of general, finite-state Markov processes. The probabilistic characterization of several properties of the reference model are discussed in Section 6 and conclusions are drawn in Section 7.

2 Overview of related works

2.1 Mathematical analysis of swarm robotics systems

Since mathematical modelling of multi-robot systems is a very complex issue, roboticists have often opted for simulations and experiments to guide their research. The complexity arising from the interactions of the individuals, the heterogeneity of the environment and the drastic simplifications required in mathematical models are among the most discussed obstacles on the way to a rigorous analysis of swarm systems.

Several theoretical models (mostly using continuum limit generalizations) have been nonetheless proposed in the literature. An extensive review of methods and techniques from a swarm engineering perspective is given in [24]. This section lists only a short number of publications that are relevant to the objectives of the paper, namely to develop probabilistic models for certain macroscopic characteristics of a swarm of robots.

In this context, models for the characterization of collective behaviours like aggregation, dispersion and foraging have been proposed in [22, 26, 42, 47]. The well known approach proposed in [26] is based on the so-called *rate equation* model: a series of coupled differential equations describing the dynamics of the swarm.

Spatial and non-spatial models for self-organization in biological systems are discussed in a series of articles [4, 8, 17, 20, 21, 31, 34, 45]. Other specialized models have been subsequently proposed: [47] presented a promising formal method using temporal logic to specify emergent swarm behaviours. An analytical model for aggregation has been derived by applying combinatorics and linear algebra in [42], while [13] follows the modelling approach originating in physics and uses Brownian motion and Fokker-Planck equations for the study of simple swarm robotics systems.

More recent models for the analysis of agent-based opinion dynamics in swarm robotics using concepts of dynamical systems theory have been proposed [6, 40].

An important contribution is given in [3] with a comprehensive approach to model swarm robotics systems by using analogies from the domain of chemical reaction modelling. Finally, quantitative analysis of important properties of swarm robotics systems as well as the modelling and validations of complex scenarii based on formal languages are discussed in recent works such as [12] and references therein.

The majority of these works are focused on long-term behaviour and convergence time of systems under mean-field approximations.

2.2 Brief review of the literature on integrals of Markov processes

Due to its relevance in numerous practical applications, the problem of evaluating integrals of Markov processes has been quite extensively studied outside the swarm robotics literature. An early result introduced in [16] presents a methodology for evaluating the expected value of the integral under the stochastic path of a birth-death process as well as the expected time to extinction for different kinds of processes.

In the case of birth-death processes, the expression of the solution is based on a system of recursive linear equations that can be solved analytically only in simple but useful cases [16].

The simplifications introduced in early works have been removed in more recent results obtained

in the context of potential theory [33, 46]. In particular, Pollett and Stefanov [37, 38] generalized the approaches of [16] and [44] to a wider variety of models and removed restrictive assumptions. Moreover, since the method of proof adopted no longer relies on state-space truncations, cases that had been previously overlooked could be studied.
In the case of birth-death processes, the Laplace transform of the distributions of first passage times and other important characteristics have been obtained [2, 9, 38]. Despite the relative simplicity of writing the resolving equations, however, it can be remarkably difficult to obtain closed form solutions for the distributions of interests.
In a recent paper, Crawford and Suchard [5] proposed an efficient error-controlled algorithm for computing transition probabilities for general birth-death processes. This new promising method constitutes a robust computational tool to obtain finite-time transition probabilities for general birth-death processes which are linked to the expected value of the process integral over arbitrary functions.

3 Model

This section introduces the reference scenario that will be used throughout the rest of the paper to illustrate the theoretical results. Although the results described in Sections 4 and 5 are general enough to be applied to non-linear Markov chains, the derivation of estimators requires the solution of rather complex systems. For the sake of simplicity, we consider robots whose activation follows a linear birth and death dynamics. This choice allows us to focus on well established closed-form solutions to the inversion of certain Laplace transforms for the characterization of relevant properties of the system.
The scenario is as follows: at time $t = 0$, a large number of robots are in a region called *nest* while an initial number of robots denoted by $X(0)$ is engaged in a certain task, and classified as active.
Active robots decide independently when to stop the execution of the task. When they stop, they return to the nest. The decision to return to the nest can be triggered by an internal stimulus, such as a low battery level, or by an external event (such as the finding of an object). Independently of the cause of the action, we assume that the rate at which robots return to the nest is known or measurable and has a value of μ_n, where n denotes the actual number of active robots. Similarly, robots in the nest decide to become active at a rate denoted by λ_n. For linear BDPs, the more robots are active, the higher is the rate at which robots leave and return to the nest. More particularly, $\mu_n = n\mu$ and $\lambda_n = n\lambda$.
The mechanism causing this dynamics is not fundamental to the discussion. It can be based for instance on direct messaging (active robots send messages to the nest to trigger the activation of other robots) or based on a pheromone-like exchange mechanism in the nest such that robots are stimulated to activate faster when the level of pheromone in the nest decreases because of the robots that have already left.
In this simple scenario, the number of active robots at time t can be modelled as a continuous-time Markov process $X(t)$ on $\{0, 1, \ldots\}$ with transition rates:

$$\mu_n = n\mu, \quad \text{if} \quad n \geq 1 \tag{1}$$

$$\lambda_n = n\lambda, \quad \text{if} \quad n \geq 0 \tag{2}$$

Figure 1 shows the state diagram of the resulting birth-death process. Figure 2 illustrates a possible realization of the process over a positive function $f(x)$, $x \in \{0, 1, \ldots\}$. For the sake of visualization, the function is represented as continuous. The values of the function $f(x)$ over the states of the chain are represented by levels of grey and can be thought as the amount of energy consumed by a swarm of size x (white being higher values).
The integral of the resulting process can be associated to the amount of energy consumed by the swarm from time $t = 0$ until the complete deactivation of all the robots. Similarly, depending

on the meaning of the function $f(x)$, other properties such as the average number of objects retrieved in a foraging scenario, the average force exerted on an object or the average area patrolled by the swarm, can be expressed as the integral of the stochastic process governing the dynamics of the robots.

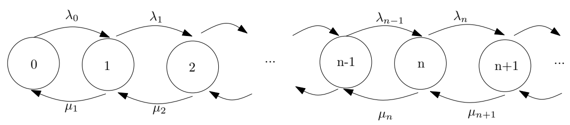

Figure 1: State diagram of a birth-death processes with rates λ_n and μ_n.

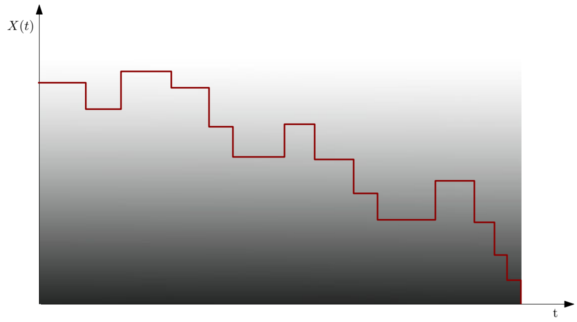

Figure 2: A possible realization of the birth-death process over a linear positive function.

In the scenario outlined above, we assume that a large number of robots are available in the nest.

4 Notation and basic results

Formally, a general birth-death process is a continuous-time Markov process $\mathcal{X} = \{X(t), t \geq 0\}$ counting the number of arbitrarily defined elements in existence at time $t \geq 0$. The process is characterized by non-negative instantaneous birth rates λ_n and death rates μ_n with $n \geq 0$. These rates are time-homogeneous constants but they may depend on the number n of elements. The instantaneous transition of the process occurs from state n either to state $n-1$ or $n+1$. Let $X(0) = k$ for some $k > 0$ and τ_k denote the first time the birth-death process reaches state 0 starting from state k:

$$\tau_k = \inf\{t > 0 : X(t) = 0 | X(0) = k\} \tag{3}$$

We are interested in integrals of a function of $X(t)$ up to the extinction time:

$$Y_f(k) = \int_0^{\tau_k} f(X(t))dt \tag{4}$$

When $f(x) = x$ we will use the notation $Y(k) = Y_f(k)$, while $Y_f^{\infty}(k)$ denotes $Y_f(k)$ when the integration limit is set to infinity. The physical meaning of this integral obviously depends on the integrand function. As noted before, in a system where $f(x)$ represents the energy consumption of a swarm of size x in a unit of time, and the robots activation and deactivation can be modelled as a birth-death process, $Y_f(k)$ represents the energy consumption of the system until the deactivation of all the robots. Similarly, in a foraging scenario where $f(x)$ represents the average number of objects collected by a swarm of size x, $\mathbb{E}(Y(k))$ represents the expected result of the foraging activity.

Hernández-Suárez and Castillo-Chavez discuss a basic derivation of $\mathbb{E}(Y(k))$ based on the study of the modified stationary distribution for different birth-death Markov processes [16]. Since their approach clarifies the structure of the problem, it is resumed here.

The solution is derived by substituting the transitions to state 0 with transitions to the initial state k. Whenever the resulting process is ergodic, a stationary distribution exists. If S_n denotes the random vector of the total amount of time spent in state n before state 0 is reached, it is straightforward to see that:

$$Y(k) = \int_0^{\tau_k} X(t)dt = \sum_n nS_n \tag{5}$$

The expected time spent in an arbitrary state n during a cycle, and the time spent in state n in the j-th cycle of the modified process are denoted by $\mathbb{E}(S_n)$ and S_{nj} respectively. By the strong Law of Large Numbers:

$$\pi_n = \lim_{r\to\infty} \frac{\sum_{j=1}^{r} S_{nj}}{\sum_{j=1}^{r}\sum_i S_{ij}} = \frac{\mathbb{E}(S_n)}{\mathbb{E}(\sum_i S_i)} \tag{6}$$

where π_n is the expected proportion of time the process spends in state n before reaching state 0. Since $\mathbb{E}(\sum_i S_i) = \mathbb{E}(\tau_k)$, and $\mathbb{E}(S_1) = \mu_1^{-1}$ (see [16], Appendix A), it follows that $\mathbb{E}(\tau_k) = (\pi_1\mu_1)^{-1}$ and:

$$\mathbb{E}(S_n) = \pi_n \mathbb{E}(\tau_k) \tag{7}$$

From Eq.(5):

$$\mathbb{E}(Y(k)) = \sum_n n\mathbb{E}(S_n) = \mathbb{E}(\tau_k)\sum_n n\pi_n \tag{8}$$

As noted in [16] the method can be used to calculate the expected time for the process to reach any element in an arbitrary set by changing transition to elements in the set with transitions to the initial state k.
For the birth-death processes that allow for the analytical expression of π_n, Equation (8) can be applied straightforwardly. Birth-death processes with inital state $k = 1$, have quasi-stationary distribution $\pi_n = H_1(n)\pi_1$, where:

$$H_j(i) = \frac{\lambda_j\lambda_{j+1}\lambda_{j+2}\dots\lambda_{i-1}}{\mu_{j+1}\mu_{j+2}\dots\mu_i}, \qquad 1 \le j < i \tag{9}$$

with the convention that $H_i(i) = 1$, $i = \{1, 2, \dots\}$, $H_j(i) = 0$ if $i < j$. By direct application of Eq.(8) it follows:

$$\mathbb{E}(Y(1)) = \frac{1}{\pi_1\mu_1}\sum_{i=1}^{\infty} iH_1(i)\pi_i \tag{10}$$

The generalization to the case $k > 1$ is based on the solution of the system of linear equations:

$$\pi_n\mu_n = \pi_{n-1}\lambda_{n-1} + \pi_1\mu_1, \qquad n \le k \tag{11}$$

$$\pi_n\mu_n = \pi_{n-1}\lambda_{n-1}, \qquad n > k \tag{12}$$

Explicit solutions for different birth-death problems are presented in [16]. However, because of errors in the paper (as pointed out by [44]), care has to be taken in the application of the formulae.

The results of Hernández-Suárez and Castillo-Chavez offer an insight on the structure of the problem but are of limited practical use. A method of proof that relies on a state-space truncation argument is discussed in [44] and generalizes these results. When the integrand function is taken into account, similarly to (5) one can write:

$$Y_f(k) = \sum_{i=1}^{\infty} f(i)S_i(k) \tag{13}$$

where $S_i(k)$ denotes the total time spent in state i within the random time interval $[0, \tau_k]$. Denote by τ_k^m, $Y^m(k)$ and $S^m(k)$ the counterparts of τ_k, $Y(k)$ and $S(k)$ where the state space S^m is truncated at $m \ge k$. In this case:

$$\mathbb{E}(Y_f^m(k)) = \mathbb{E}\left(\sum_{i=1}^{m} f(i)S_i^m(k)\right) = \sum_{j=1}^{k}\frac{1}{\mu_j}\sum_{i=j}^{m} f(i)H_j(i) \tag{14}$$

where the last equality is due to the following result discussed in [43]: $\mathbb{E}(S_i^m(k)) = \sum_{j=1}^{k} \frac{H_j(i)}{\mu_j}$ When the state space is not finite, Eq.(14) can be generalized by noting that (from the monotone convergence theorem):

$$\mathbb{E}(Y_f(k)) = lim_{m\to\infty}\mathbb{E}\left(\sum_{i=1}^{m} f(i)S_i(k)\right) \tag{15}$$

and that $\mathbb{E}(Y_f^m(k)) \to \mathbb{E}(Y_f(k))$ when $m \to \infty$ as long as

$$\mathbb{E}\left(\sum_{i=1}^{m} f(i)S_i(k)\right) \leq \mathbb{E}(Y_f^m(k)) \leq \mathbb{E}(Y_f(k)) \tag{16}$$

The first inequality is straigthforward, while for the second inequality to hold it is necessary to require that a positive real number x_0 exists, such that f is non-decreasing for $x \geq x_0$ [44]. This assumption can be removed by using results of potential theory [37]. In view of (14) and (16) the following holds:

Proposition 4.1 ***[Stefanov-Whang]*** *The expected value of $Y_f(k)$ where the birth and death rates are λ_i and μ_i has the closed form expression:*

$$\mathbb{E}(Y_f(k)) = \sum_{j=1}^{k}\left(\frac{1}{\mu_j}\sum_{i=j}^{\infty} f(i)H_j(i)\right) \tag{17}$$

Proposition 4.1 is based on the assumption that the process is non-explosive. The generalization of equation (17) is given in [37] and discussed in the next section.

5 Path integrals for general continuous-time Markov chains

In a series of articles, Pollett et al. [37, 38] generalize the results introduced in the previous section by using theorems developed in the context of potential theory.
In his works, Pollett provides a complete analysis of the birth-death processes studied by Stefanov and Whang in [44] as well as an extension to birth-death-catastrophe processes. This section resumes the results discussed in [37]. The work is particularly relevant since not only it removes the constraints imposed in [44], but also clarifies some technicalities and extends the analysis to cases that had not been previously addressed.

5.1 Expectation of the path integral

The transition rates of a continuous Markov chain are typically expressed in matrix form. The so-called q-matrix is a fundamental descriptor of the process from which important properties can be derived. A discussion of q-matrices and their applications can be found in [33]. In the following, the elements of the q-matrix associated to the Markov chain (assumed to be stable and conservative) of the process are denoted by q_{ij}. They represent the transition rate from state i to state j for $j \neq i$. Moreover, $q_{ii} = -q_i := -\sum_{j\neq i} q_{ij}(< \infty)$ represents the total rate out of state i. In the case of a birth-death process, the non null elements of the transition matrix are $q_{i,i+1} = \lambda_i$ and $q_{i,i-1} = \mu_i$ with $\mu_0 = 0$.
Let $S = \{0, 1, \ldots\}$ be the set of non-negative integers where the continuous Markov chain is defined. The following theorem is a well known result of potential theory [37, 33]:

Proposition 5.1 *If $y_i = \mathbb{E}(Y_f^\infty(i))$, then $y = (y_i, i \in S)$ is the minimal non-negative solution to the system of equations*

$$\sum_{j\in S} q_{ij}z_j + f_i = 0, \qquad i \in S \tag{18}$$

in the sense that y satisfies these equations, and, if $z = (z_i, i \in S)$ is any nonnegative solution, then $y_i \leq z_i$ for all $i \in S$

Proposition 5.1 can be applied to the problem of estimating the path integral of Eq.(4) up to its absorption time by setting $q_0 = 0$ and $q_{0j} = 0$ for $j \geq 1$ and $f(0) = 0$. In this case it can be restated as follows [37, 33]:

Corollary 5.2 *If $e_i = \mathbb{E}(Y_f(i))$, where $Y_f(i)$ is given by (4), then $e = (e_i, i \geq 1)$ is the minimal non-negative solution to the system of equations*

$$\sum_{j\geq 1} q_{ij} z_j + f_i = 0, \qquad i \geq 1 \tag{19}$$

These results are relevant in the case of swarm robotics applications that can be modelled as a time-continuous Markov chain and where the characterization of the expected value of integrals of type (4) is meaningful. For birth-death processes explicit solutions to the system (18) have the following form [37]:

Proposition 5.3 *For birth-death processes,*

$$\mathbb{E}(Y_f^{\infty}(j)) = \sum_{i=j}^{\infty} \frac{1}{\lambda_i H_1(i)} \sum_{k=0}^{i} f(k) H_1(k) \tag{20}$$

While the solutions to the system (19) under the assumption that the process is non-explosive have the following form [37]:

Proposition 5.4 *[Pollett] For the birth-death processes with $\sum_{i=1}^{\infty} \frac{1}{\mu_i \pi_i} = \infty$:*

$$\mathbb{E}(Y_f(j)) = \sum_{i=1}^{j} \frac{1}{\mu_i \pi_i} \sum_{k=i}^{\infty} f(k) \pi_k \tag{21}$$

with the potential coefficients given by $\pi_1 = 1$ and $\pi_k = \prod_{j=2}^{k} \frac{\lambda_{j-1}}{\mu_j}$ which is finite if $\sum_{k=i}^{\infty} f(k)\pi_k < \infty$

It is easy to see that Eq.(21) is equivalent to the result derived by Stefanov-Whang in Eq.(17), since:

$$\begin{aligned}
\sum_{i=1}^{k} \frac{1}{\mu_i \pi_i} \sum_{j=i}^{\infty} f(j)\pi_j &= \sum_{i=1}^{k} \frac{1}{\mu_i H_1(i)} \sum_{j=i}^{\infty} f(j) H_1(j) \\
&= \sum_{i=1}^{k} \frac{1}{\mu_i} \sum_{j=i}^{\infty} f(j) \frac{H_1(j)}{H_1(i)} \\
&= \sum_{i=1}^{k} \frac{1}{\mu_i} \sum_{j=i}^{\infty} f(j) H_i(j)
\end{aligned}$$

Figures 3 and 4 show the expected value of the integral of a birth-death process with rates $\lambda_i = 0.5$ and $\mu_i = 1$ and different initial values over two different functions. The closed form solution is compared with the average obtained with a Montecarlo simulation over 1000 runs.

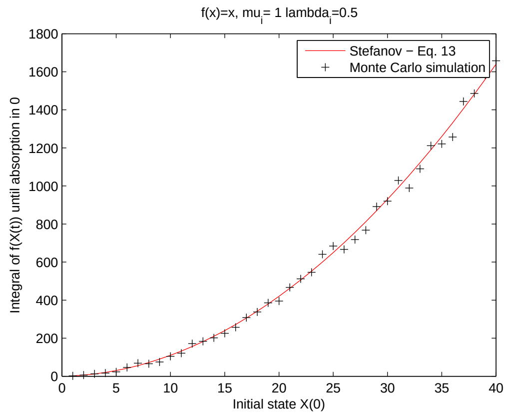

Figure 3: Expected value of the integral of a birth death process with $\lambda_i = 0.5$ and $\mu_i = 1$ over the function $f(x) = x$.

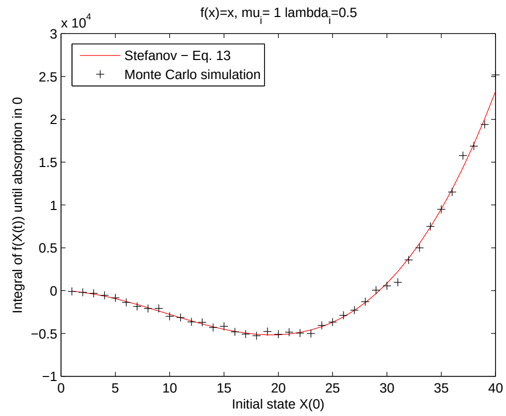

Figure 4: Expected value of the integral of a birth death process with $\lambda_i = 0.5$ and $\mu_i = 1$ over the function $f(x) = 2x^2 - 40x$

5.2 Distribution of the path integral

Extensions of the propositions 5.1 and 5.2 allow for the characterization of the Laplace transform of the distribution of the path integral of Eq.(4). A method for obtaining the Laplace transform of the distribution of $Y_f(k)$ is described in [38, 46] and demonstrated on simple birth and death processes for which an explicit form of the distribution can be obtained.
The method is based on the following extension of Proposition 5.1 where $\tau_k = \inf\{t \geq 0 : X(t) \notin A\}$ with A a fixed subset of the state space $S = \{0, 1, \ldots\}$ and $f : A \to [0, \infty)$.

Proposition 5.5 *Let $y_i(\theta) = \mathbb{E}(e^{-\theta Y_f(i)})$. For each $\theta \geq 0$, $y(\theta) = (y_i(\theta), i \in S)$ is the maximal solution to the system:*

$$\sum_{j \in S} q_{ij} z_j = \theta f_i z_i, \qquad i \in A \tag{22}$$

with $0 \leq z_j \leq 1$ for $j \in A$ and $z_j = 1$ for $j \notin A$, in the sense that $y(\theta)$ satisfies these equations, and, if $z = (z_i, i \in S)$ is any such solution, then $y_i(\theta) \geq z_i$ for all $i \in S$.

In particular, the case $f_i = 1, \forall i$ corresponds to the problem of determining the distribution of first passage times of the Markov chain, that is, the distribution of the time it takes the chain to exit from the set A.
In the case of birth death processes results on the transforms of the distribution of first passage times have been obtained by Flajolet, Guillemin, Stefanov and Ball in terms of continued fractions [9, 2]. By using proposition 5.5, Pollett and Stefanov derive in [38] the distribution of $Y(k)$ whose Laplace transform can be inverted.
As noted by McNeil [28], the distribution of these kind of integrals is related to the distribution of the hitting times of modified Markov chains (details can be found in [38, 28]). The derivation of closed form solutions, however can be very complicated even in relatively simple birth-death processes.
In a pioneering series of articles Karlin and McGregor [19] developed a formal theory of general birth-death processes that allows for the expression of transition probabilities in terms of a sequence of orthogonal polynomial and a spectral measure. In the case of general birth and death rates, however, no clear method to derive the orthogonal polynomials is known. Additionally, the transition probability often does not possess an analytic representation or a convenient computational form. Indeed, closed form solutions for the transition probabilities of general birth-death processes are known only for few types of processes.

Given the difficulty in obtaining closed form expressions for the transition probabilities of general birth-death processes, several authors have made progresses in designing approximate numerical methods to compute and invert the continued fraction representation of their Laplace transform [30]. In a recent remarkable article, Crawford and Suchard introduce an error-controlled algorithm for computing these transition probabilities and for analysing the approximation error [5]. To the best of our knowledge, the application of these novel techniques to the approximation of the distribution of path integrals has not been investigated.

6 Applications in Swarm Robotics

In this section we exploit known analytical expressions of the solutions to the system 5.2 for path integrals associated to the behaviour of the swarm robotics described in Sec.3. It is assumed that the model exhibits extinction (i.e., all the robots will eventually return to the nest) hence $\lambda < \mu$.
The expected value and distribution of several quantities are calculated and tested against Monte Carlo simulations.
First, we will consider the so called time-to-extinction, i.e., the time it takes to the process to reach the state $X(t) = 0$, starting from an arbitrary state $X(0) = m$. By knowing the probability density function of the random variable associated to this time it is possible to measure the probability that the swarm is still active after a certain time, and if necessary, to modify the parameters in order to obtain the desired behaviour.
For the same process, the probabilistic characterization of the power consumption of the system from time $t = 0$ to the time corresponding to the deactivation of the last robot is given under the assumption that each robot has the same consumption profile. Finally the amount of work performed by the swarm in case of spatial interferences is considered, as well as the cost-benefits of particular solutions.

6.1 Time to extinction

By setting $f(x) = 1$, Eq.(4) represents the total activity duration of the swarm, from time $t = 0$ to the deactivation of the last robot. The distribution of $Y_1(k)$ can be obtained by using Proposition 5.5. For linear birth-death processes, however, closed-form solutions for the passage time probabilities $P_{m,n}(t) = Pr(X(t) = n|X(0) = m)$ are available (Bailey 1964):

$$P_{m,n}(t) = \sum_{j=0}^{\min(m,n)} \binom{m}{j} \binom{m+n-j-1}{m-1} \alpha^{m-j} \beta^{n-j} (1-\alpha-\beta)^j \tag{23}$$

where:

$$\alpha = \frac{\mu(e^{(\lambda-\mu)t} - 1)}{\lambda e^{(\lambda-\mu)t} - \mu} \qquad \text{and} \qquad \beta = \frac{\lambda(e^{(\lambda-\mu)t} - 1)}{\lambda e^{(\lambda-\mu)t} - \mu} \tag{24}$$

The absorption time probability from an intial state m to the absorbing state 0 is thus:

$$P_{m,0}(t) = \left(\frac{\mu(e^{(\lambda-\mu)t} - 1)}{\lambda e^{(\lambda-\mu)t} - \mu} \right)^m \tag{25}$$

The probability density function of the activity time is obtained from Eq.(25):

$$f_{m,0}(t) = \frac{m\mu(e^{(\lambda-\mu)t} - 1)^{m-1} \mu(\lambda-\mu)^2 e^{(\lambda-\mu)t}}{(\lambda e^{(\lambda-\mu)t} - \mu)^{m+1}} \tag{26}$$

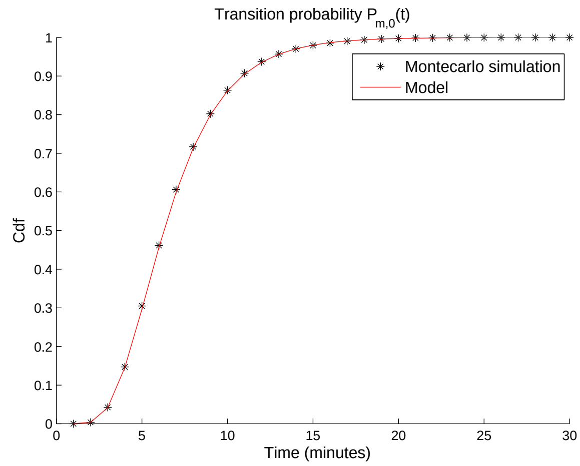

Figure 5: Cumulative distribution function of the activity times of a swarm modelled as in Sec.3. Initial number of robots $X(0) = 20$, $\lambda = 0.6$, $\mu = 1$.

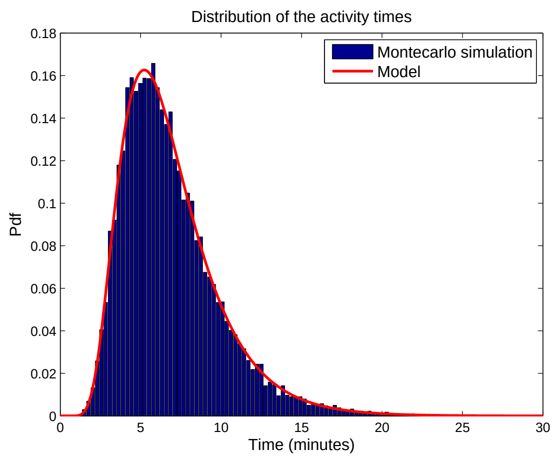

Figure 6: Probability density function of the activity times of a swarm modelled as in Sec.3 as predicted by Eq.(26) and Montecarlo simulation of (15000 runs). Initial number of robots $X(0) = 20$, $\lambda = 0.6$, $\mu = 1$.

Figures 5 and 6 show the cumulative distribution function and the probability density function of the activity time of the system described in section 3 as well as numerical results obtained via Montecarlo simulation.
The availability of these closed form solutions allows for the derivation of the parameters associated to the probabilities that robots are still active after a certain time if their dynamics follows a birth death-process.
In robotics systems where strong temporal requirements have to be respected, as for instance applications where robots should all be in a safe region after a certain deadline, the probability associated to the respect of the deadline can be calculated precisely assuming that their dynamics can be modelled as discussed above. Once the parameters associated to the physical constraints (such as the number of robots available) and to the levels of confidence that can be tolerated have been calculated, individual robotics controller can be implemented with these reference parameters in mind.

6.2 Estimation of energy consumption

Since the simplifications introduced by the model of Sec.3 are quite drastic, the expected value of the number of active robots at time t can be obtained by solving the differential equation:

$$\mathbb{E}(X(t)) = X(0)e^{(\lambda-\mu)t} \tag{27}$$

The integral of Eq.(27) multiplied by the unitary consumption provides the expected value of the energy consumed by the system.
When the birth and death rates are not linear on the number of active robots, Eq.(17) and Eq.(20) provide a faster alternative to the solution of the first-order non-linear ordinary differential equation governing the expected amount of energy consumed by the system. Figures 7 and 8 show two realizations of the process as well as the expected consumption. When strict requirements about energy consumption have to be respected, the knowledge of the expected value is not sufficient. In this case, proposition 5.5 can be used to obtain the distribution of the energy consumed by the system $\mathcal{E} = \int_0^t cX(t)dt$.
In case of linear birth-death-processes, and energy consumption per time unit $c = 1$, the system

of Eq.(22) can be solved explicitly and the Lapace transform can be inverted to obtain the probability density function [37]:

$$dP_i(\mathcal{E} \leq x) = \frac{i}{x} e^{-(\lambda+\mu)x} (\mu/\lambda)^{i/2} I_i(2x\sqrt{\lambda\mu}) dx \tag{28}$$

where $I_i(z)$ is the modified Bessel function of the first kind [1]. Figures 9 shows the probability density function of the energy consumption associated to the robots dynamics described in Sec.3 with parameters $\lambda = 0.6$ and $\mu = 1$ along with the result of a Montecarlo simulation. The initial number of robots is set to $N(0) = 20$. Figure 10 illustrates the probability density function and the Montecarlo simulation of the energy consumed by the swarm when robots use $c = 3$ units of energy per time unit.

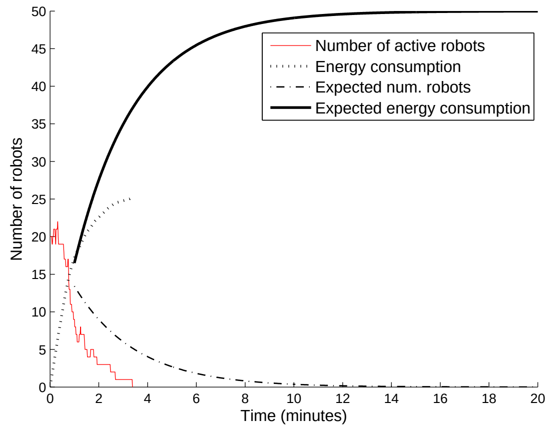

Figure 7: Expected value of the energy consumed by the system (black line) and a realization of the process resulting in half of the expected energy consumption.

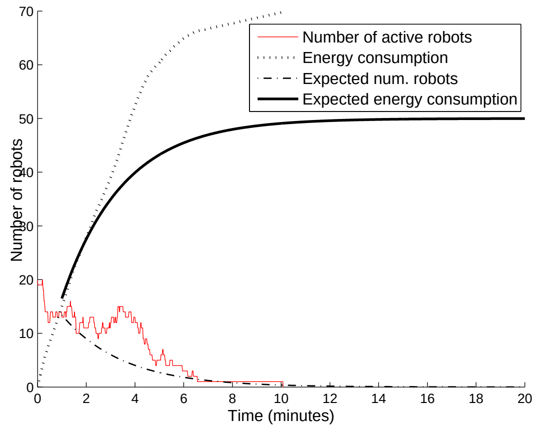

Figure 8: Expected value of the energy consumed by the system (black line) and a realization of the process resulting in a consumption 28% higher that the expected.

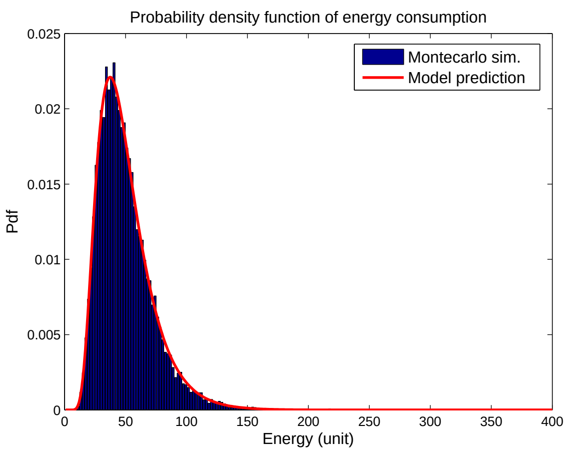

Figure 9: Probability density function of the system energy consumption until the last robot is active. $N(0) = 20$, $\lambda = 0.6$, $\mu = 1$. Single robot energy consumption per time unit $c = 1$. Model prediction and Montecarlo simulation. (15000 runs)

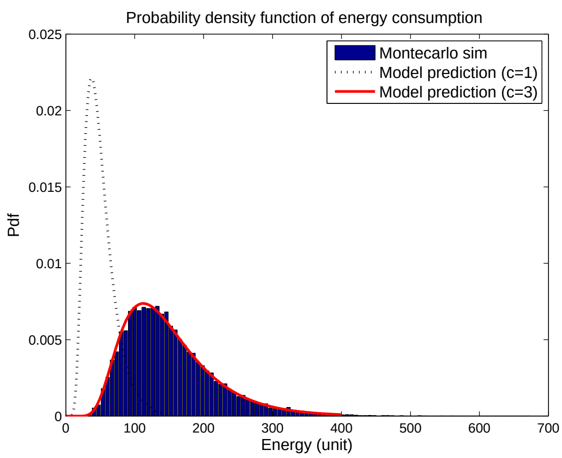

Figure 10: Probability density function of the system energy consumption until the last robot is active. $N(0) = 20$, $\lambda = 0.6$, $\mu = 1$. Single robot energy consumption per time unit $c = 3$. Model prediction compared to the case $c = 1$ and Montecarlo simulation. (15000 runs)

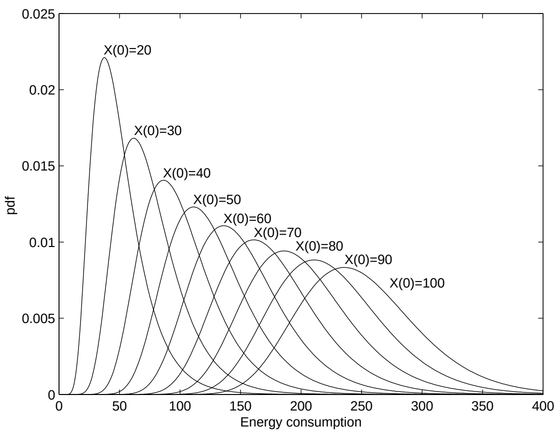

Figure 11: Probability density functions of total energy consumption with different initial swarm sizes.

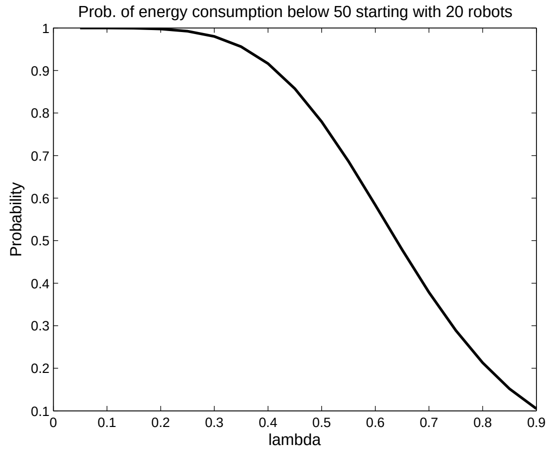

Figure 12: Probability that the total energy consumption is below a threshold of 50 units as function of the activation rate λ (assuming $\mu = 1$).

Figure 11 reports the probability density functions of the energy consumption with swarms of different initial size. When the modelling of the robot controllers on the basis of the birth-death process described in Sec.3 is possible, Eq.(28) can be employed for the determination of the parameters that guarantee an energy consumption below critical levels with a certain probability. Figure 12, for instance, shows the probability that the energy consumed by the system is below a threshold of 50 units as a function of the activation rate λ (assuming $\mu = 1$).

6.3 Estimation of the amount of work performed in presence of spatial interferences

The amount of work performed by a swarm depends on the number of active robots at each time step. This dependence is generally complex and non-linear as the competition for resources (physical space, energy, communication channels) can significantly influence the behaviour and the performance of the robots.

The effect of interferences on the global performance of a swarm robotics system has long been recognized as an important issue in multi-robot systems design. Its analysis leads, in general, to two alternative situations: in some applications the performance of the swarm improves (generally sub-linearly) as the number of robots increases, in others an optimal group size associated to the peak performance exists. In this second situation, the increase in the number of robots above the optimal value causes the performance to decrease and eventually to reach zero. On the other hand, the performance of each robot tends to decrease when the number of active robots increases (in scenarios without cooperation) [22].

Approaches to understand quantitatively and to minimize the effect of interferences have been discussed in [35, 22] and references therein.

These works generally focus on finding the optimal swarm size that maximizes the performance of the system for a given task and a given controller [32, 10, 22, 35]. The problem is addressed with simulations or analytical techniques in order to characterize the optimal group size. We refer in this case to the *static* version of the problem, in the sense that once the optimal number of robots is found, the swarm is configured with that number and the size does not change. In many applications, however, the swarm size changes as the robots stop working or move to different areas.

In this section we study a *dynamical* version of the problem with the goal of estimating the expected amount of work accomplished by a swarm of robots with the birth-death dynamics introduced in section 3 in presence of spatial interferences. Similarly to the works of Lerman et al.[22] we consider a simplified robot controller composed of two states (Figure 13): a working state and a state corresponding to the management of the spatial interference (i.e., obstacle avoidance).
It is assumed that robots are distributed over a closed area, with an activity to perform (for example cleaning, cutting lawn, patrolling). The activity does not require cooperation. Furthermore, a robot in obstacle-avoidance mode suspends its main activity for all the time necessary to maneuver.

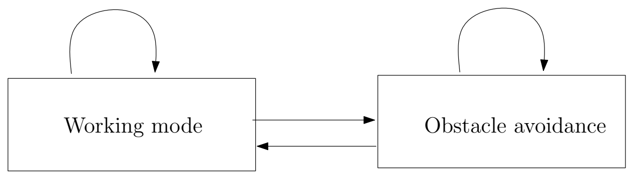

Figure 13: Simplified robot controller and transitions between the states.

Mathematical models for different kind of interferences have been discussed in [22, 14]. However, interferences represent a complex phenomenon that depends not only on the physical characteristics of the robots but also on the kind of tasks, on the geometry of the environment and on the strategy adopted by the robots. For this reasons, we consider computer simulations more accurate than mathematical models in capturing the subtleties of the phenomenon and for the determination of performance functions. In the following, the simulator ARGoS [36] is used to characterize the effect of spatial interferences by calculating an average performance function via repeated high-level simulations. ARGoS is a discrete-time physics-based simulation framework developed within the Swarmanoid[1] project. ARGoS can simulate various robots at different levels of detail, as well as a large set of sensors and actuators; the experiments presented in this work are carried out in a 2-dimensional kinematics-based simulation. The measurements provided by ARGoS are used to study the expected amount of work performed by a swarm of robots under different conditions.
In particular, robots are distributed in a $4m \times 4m$ arena, Figure 14, and are equipped with 24 proximity sensors distributed around their perimeter. The robots are able to move at 5 cm/s, Figure 15. The performance of the swarm is measured by the average area they are able to cover in a time unit, in working mode. Whenever the robot senses a close obstacle within its front sensing area, it stops executing the task and turns in the direction of the left or right areas. In the case, front, left and right areas are blocked by obstacles, the robot reverses its motion and move backwards, if possible. In the simulations, the performance of the system is measured over an interval of 1000 seconds. Several measurement are collected during a simulation:

- The average area covered by a robot in working mode during the simulation.
- The average area covered by all the robots in working mode during the simulation.
- The average time spent by the robots in working mode

Collision avoidance maneuvers cause the performance to decrease when the number of robots increases. Figure 17 shows that the swarm performance increases by increasing the number of robots until an optimal value is reached. Additional robots cause the performance of the system to decrease until the zero value is reached. Figures 14 and 15 illustrate two snapshots of a simulation performed in ARGoS. The effect of spatial interferences reduces the expected amount of work since robots in obstacle-avoidance mode suspend their main activity.

[1] http://www.swarmanoid.org/

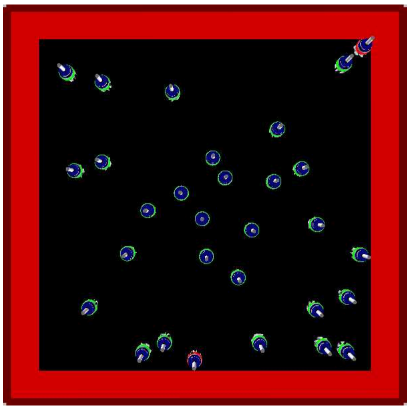

Figure 14: Robots distributed in the arena at time $t = 0$. ARGoS simulation.

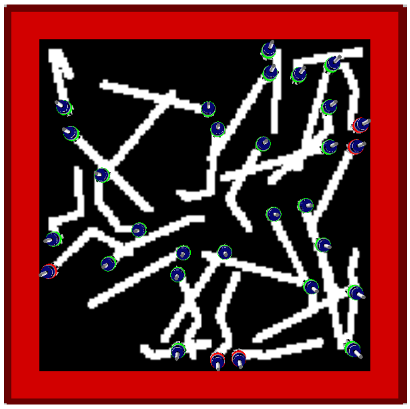

Figure 15: The amount of work performed visualized as the regions of the space that have been visited. ARGoS simulation.

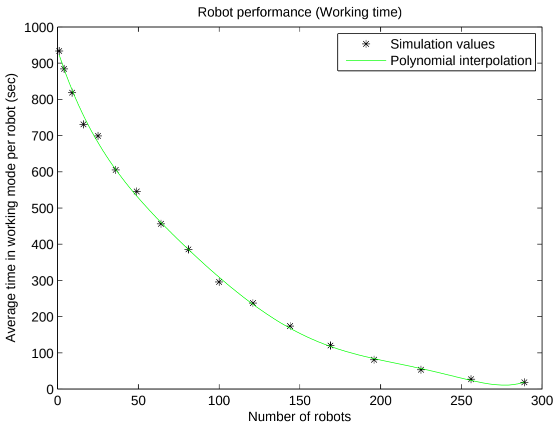

Figure 16: Efficiency of a single robot operating in swarms of different sizes defined as the average amount of time spent in working mode in a simulation of 1000 seconds. Data are collected with ARGoS simulator and interpolated with a polynomial of degree 6.

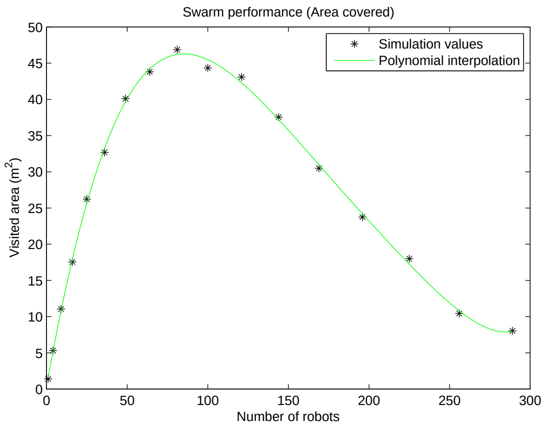

Figure 17: Efficiency of the swarm measured as the average area covered by the robots in working mode during a simulation of 1000 seconds. Data are collected with ARGoS simulator and interpolated with a polynomial of degree 6.

The negative effect of spatial interference on the performance of the system is shown in Figures 16 and 17 reporting the average time a single robot spends in working mode and the average area covered by the swarm as measured by ARGoS. Data are interpolated with a polynomial of degree 6.

When the dynamics of the activation and deactivation of robots is modelled as the birth-death process described in Section 3, the value of integral (4) over the interference function corresponds to the expected amount of work performed by the swarm. This value is calculated by using as interference function the polynomial interpolation of the data provided by the simulator divided by the duration of the simulation.

Figures 18 and 19 report the expected value of the integral (4) as computed by Eq.(17), when $\lambda = 0.6$ and $\mu = 1$. In particular, Fig.18 reveals how the effect of spatial interferences puts an upper bound on the expected output of the system. In absence of interference phenomena, the output of the system would be linear with the number of robots, as shown by the dashed line in Fig.18. When interference phenomena are taken into account, the model correctly predicts the expected value of the integral. It seems that a neat closed form expression for the variance is not derivable. Nonetheless, a route for its numerical evaluation linked to the resolution of a large system of equations is described in [44]. Figure 19 shows the temporal dynamics of the expected amount of work with different swarm sizes obtained via Montecarlo simulation over 1000 runs. The result converges to the value predicted by Eq.(17).

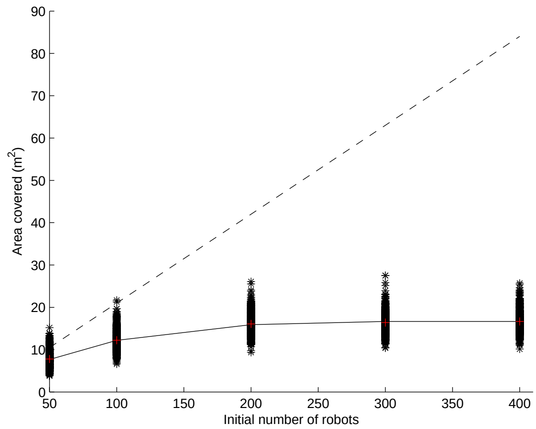

Figure 18: Expected value of the amount of work performed by swarms of varying initial size (continuous line) plotted on top of Montecarlo simulations (asterisks). The interference effects put an upper bound on the amount of work the system can perform which would be linear otherwise (dotted line).

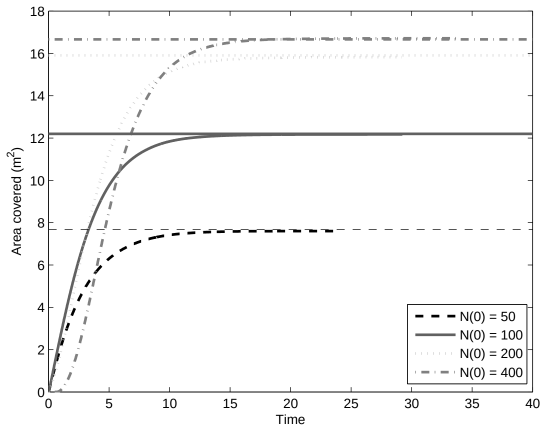

Figure 19: The effect of the interferences manifests itself in the dynamics of the expected amount of work plotted with respect to time for different initial swarm sizes.

6.4 Estimation of costs and rewards

Theoretical results on the characterization of the integral of BDPs can be used to calculate the expected cost-reward associated to swarms with different size and dynamics, whenever this concept is relevant.

In applications like foraging, for example, the cost-reward of an algorithm may be related to the energy income of the swarm over time. This, in turn, may be related to the ratio of foragers and resters (division of labour) in the group. We observe in this section the dynamics of the energy function associated to the system with the birth-death dynamics discussed in section 3. The reward function is defined as follows:

$$E(n) = R(n)\kappa - C(n) \tag{29}$$

where $C(n)$ and $R(n)$ represent the cost and the reward associated to a swarm of size n respectively, and the constant κ has the dimension of cost/reward. Equation (29) constitutes a generalization of the energy-balance function defined in [23] in the context of a foraging application where the reward of the system is related to the amount of energy the swarm is able to collect during the foraging activity and the cost function is related to the energy consumption associated to the activity of foraging.

Since the measure of the performance of a swarm system relative to its costs may be secondary to energetic consideration we prefer to talk in terms of general cost and reward functions defined on a case per case basis.

To illustrate the application of the results of Sec.4 to the computation of the expected reward, consider the cost function shown in Fig.20 defined as:

$$E(n) = f(n)\kappa - n * 0.02 \tag{30}$$

where $\kappa = 0.5m^{-2}$ and $f(n)$ corresponds to the average area covered by the robots in working mode during a minute (obtained from the function plotted in Fig.17). In this case a swarm of n robots is awarded for each unit of area robots can cover and penalized proportionally to the number of active robots. Swarms configured with an increasing initial number of robots and whose dynamics is modelled by a birth-death process with parameters $\lambda = 0.6$, $\lambda = 0.3$ and $\mu = 1$ are simulated. The expected value of the cost-reward function of Eq.(30) shown in Fig.17 is calculated by applying Eq.(17). Since in presence of spatial interferences the robots will spend on average less time in working mode, the benefits of using large swarms are rapidly eliminated by the costs. With the structure of costs and rewards introduced above and the linear birth-death dynamics of the activation of the robots, the average reward of the system becomes negative for swarms with an initial size larger than 140 robots.

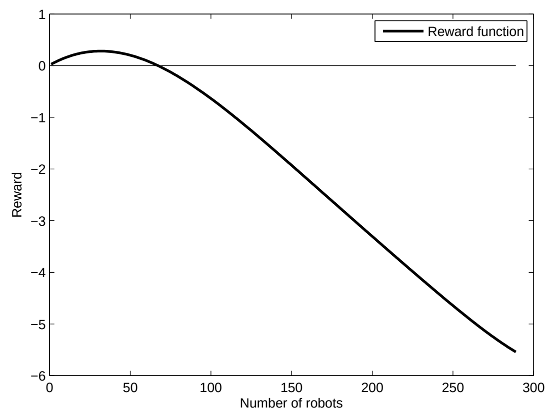

Figure 20: Cost-reward function of Eq.(30) with $\kappa = 0.5$

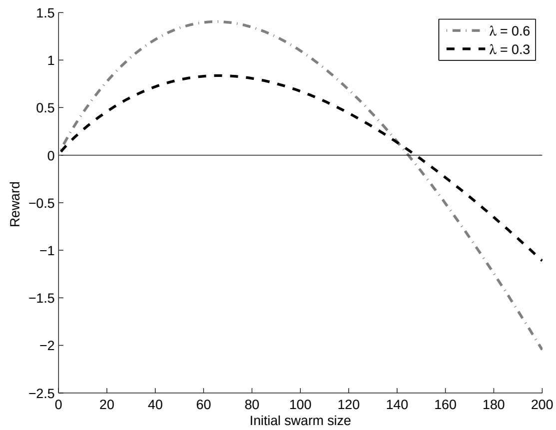

Figure 21: Expected reward of the activity of the swarm modelled as in Sec.3 calculated by using Eq.(17) for swarms of different initial size.

7 Conclusions

The mathematical modelling of swarm robotics systems is a complex task. Often, the effect of the interactions between the robots is so complex that the only viable way to study the system is to use numerical simulations. In other cases, it is possible to derive analytical macroscopic models that can be used to investigate the properties of the system in idealized conditions. Unlike microscopic models, which describe the behaviour of individual robots, analytical macroscopic models are often constituted of a set of coupled differential equations whose numerical solution provides insights into the dynamics and long term behaviour of the system.

Although this approach can provide substantial information to guide the development of robotics controllers, the finite nature of the system allows for outcomes that cannot be captured by continuum approximations. Moreover, under mean field assumptions it may not be possible to characterize probabilities associated to specific events.

In this paper we investigated the use of birth-death processes for the probabilistic modelling of distributed robotics systems. The objective is to evaluate whether theoretical results developed in the context of the analysis of continuous-time Markov chains can be applied to the study of the properties of swarm systems. The rationale is that, in several applications, the number of robots in a particular state or engaged in a particular action constitutes the main descriptor of the state of the swarm. When the evolution in time of this number can be modelled with a birth-death process, well established results can be applied to characterize complex quantities such as expectation of path integrals and mean passage times. In this context, we reviewed

several theoretical results for the evaluation of the expected value of a path integral for general birth-death processes on countable state spaces and provided references to relevant publications. To illustrate these approaches we considered a basic example where the dynamics of the activation and deactivation of robots can be modelled as a linear birth-death process and provided the characterization of the distribution of the total activity time and the expected amount of work that the system is able to perform, even in presence of spatial interferences.
Albeit this kind of modelling involves simplifications that may not be valid in complex circumstances, in applications where the simplifications are justified, theoretical and numerical tools available in the literature of birth-death processes can be effectively used to characterize relevant probabilities concerning the energy consumption, the amount of work accomplished, or the expected cost-reward of swarm robotics systems. These probabilities, in turn, can be used to guide the development of individual robotics controllers with specific performance guarantees.
The focus on tractable stochastic processes offers a double advantage: on the one hand, robotics controllers that mimic these processes inherently possess certain quantifiable properties, on the other hand, it offers valuable guidelines for the study of the characteristics of more sophisticated controllers, designed with the traditional trial and error approach, when the macro-behaviour they generate can be reconducted to well-known dynamics, such as BDPs.

References

[1] Abate, J., Kijima, M., and Whitt, W. Decompositions of the m/m/1 transition function. *Queueing Syst. Theory Appl. 9*, 3 (Oct. 1991), 323–336.

[2] Ball, F., and Stefanov, V. T. Further approaches to computing fundamental characteristics of birth-death processes. *Journal of Applied Probability 38*, 4 (2001), pp. 995–1005.

[3] Berman, S. *Abstractions, Analysis Techniques, and Synthesis of Scalable Control Strategies for Robot Swarms.* PhD dissertation, University of Pennsylvania, Department of Mechanical Engineering and Applied Mechanics, 2010.

[4] Camazine, S., Franks, N. R., Sneyd, J., Bonabeau, E., Deneubourg, J.-L., and Theraula, G. *Self-Organization in Biological Systems.* Princeton University Press, Princeton, NJ, USA, 2001.

[5] Crawford, F., and Suchard, M. Transition probabilities for general birth-death processes with applications in ecology, genetics, and evolution. *J Math Biol 65* (2012), 553–580.

[6] de Oca, M. A. M., Ferrante, E., Scheidler, A., Pinciroli, C., Birattari, M., and Dorigo, M. Majority-rule opinion dynamics with differential latency: a mechanism for self-organized collective decision-making. *Swarm Intelligence 5*, 3-4 (2011), 305–327.

[7] Dorigo, M., and Stützle, T. *Ant Colony Optimization.* Bradford Company, Scituate, MA, USA, 2004.

[8] Edelstein-Keshet, L. *Mathematical Models in Biology.* Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 2005.

[9] Flajolet, P., and Guillemin, F. The formal theory of birth-and-death processes, lattice path combinatorics and continued fractions. *Adv. in Appl. Probab. 32*, 3 (2000), 750–778.

[10] Fontan, M. S., and Mataric, M. J. A study of territoriality: The role of critical mass in adaptive task division. In *In From Animals to Animats 4: Proceedings of the Fourth International Conference of Simulation of Adaptive Behavior* (1996), MIT Press, pp. 553–561.

[11] Gani, J., and Jerwood, D. The cost of a general stochastic epidemic. *Journal of Applied Probability 9*, 2 (1972), pp. 257–269.

[12] Gjondrekaj, E., Loreti, M., Pugliese, R., Tiezzi, F., Pinciroli, C., Brambilla, M., Birattari, M., and Dorigo, M. Towards a formal verification methodology for collective robotic systems. In *Proceedings of the 14th International Conference on Formal Engineering Methods* (2012). In press.

[13] Hamann, H. *Space-Time Continuous Models of Swarm Robotic Systems: Supporting Global-to-Local Programming.* Springer-Verlag, 2010.

[14] Hamann, H. Towards swarm calculus: Universal properties of swarm performance and collective decisions. In *Swarm Intelligence*, M. Dorigo, M. Birattari, C. Blum, A. Christensen, A. Engelbrecht, R. Groß, and T. Sttzle, Eds., vol. 7461 of *Lecture Notes in Computer Science.* Springer Berlin / Heidelberg, 2012, pp. 168–179.

[15] Hereford, J. M. Analysis of a new swarm search algorithm based on trophallaxis. In *Proceedings of the IEEE Congress on Evolutionary Computation, CEC 2010, Barcelona, Spain, 18-23 July 2010* (2010), IEEE, pp. 1–8.

[16] Hernández-Suárez, M., and Castillo-Chavez, C. A basic result on the integral for birth-death markov processes. *Mathematical Biosciences 161* (1999), 95–104.

[17] Holland, O., and Melhuish, C. Stigmergy, self-organization, and sorting in collective robotics. *Artif. Life 5*, 2 (Apr. 1999), 173–202.

[18] Jerwood, D. A note on the cost of the simple epidemic. *Journal of Applied Probability 7*, 2 (1970), pp. 440–443.

[19] Karlin, S., and McGregor, J. L. The differential equations of birth-and-death processes, and the stieltjes moment problem. *Transactions of the American Mathematical Society 85*, 2 (1957), pp. 489–546.

[20] Kazadi, S., Abdul-Khaliq., A., and Goodman, R. M. On the convergence of puck clustering systems. *Robotics and Autonomous Systems 38* (2002), 93–117.

[21] Kube, C., and Bonabeau, E. Cooperative transport by ants and robots. *Robotics and Autonomous Systems 5* (2000), 85–101.

[22] Lerman, K., and Galstyan, A. Mathematical model of foraging in a group of robots: Effect of interference. *Auton. Robots 13*, 2 (Sept. 2002), 127–141.

[23] Liu, W., Winfield, A., Sa, J., Chen, J., and Dou, L. Strategies for energy optimisation in a swarm of foraging robots. In *Proceedings of the 2nd international conference on Swarm robotics* (Berlin, Heidelberg, 2007), SAB'06, Springer-Verlag, pp. 14–26.

[24] M. Brambilla, E. Ferrante, M. B. M. D. Swarm robotics: A review from the swarm engineering perspective. *Preprint submitted to Swarm Intelligence* (2012).

[25] Mangel, M., and Tier, C. Four facts every conservation biologists should know about persistence. *Ecology 75*, 3 (1994), pp. 607–614.

[26] Martinoli, A., Ijspeert, A. J., and Gambardella, L. M. A probabilistic model for understanding and comparing collective aggregation mechansims. In *Proceedings of the 5th European Conference on Advances in Artificial Life* (London, UK, UK, 1999), ECAL '99, Springer-Verlag, pp. 575–584.

[27] Mather, T. W., and Hsieh, M. A. Distributed robot ensemble control for deployment to multiple sites. In *Robotics: Science and Systems* (2011), H. F. Durrant-Whyte, N. Roy, and P. Abbeel, Eds.

[28] McNeil, D. R. Integral functionals of birth and death processes and related limiting distributions. *The Annals of Mathematical Statistics 41*, 2 (1970), pp. 480–485.

[29] Moran, P. *The Theory of Storage*. Methuen Ltd, 1959.

[30] Murhy, J. A., and O'Donohoe, M. R. Some Properties of Continued Fractions with Applications in Markov Processes. *IMA Journal of Applied Mathematics 16*, 1 (Aug. 1975), 57–71.

[31] Murray, J. D. *Mathematical Biology*, vol. 19 of *Biomathematics*. Springer, New York, 1989.

[32] Nitz, E., Arkin, R. C., and Balch, T. R. Communication of behavioral state in multi-agent retrieval tasks. In *ICRA (3)'93* (1993), pp. 588–594.

[33] Norris, J. R. *Markov Chains (Cambridge Series in Statistical and Probabilistic Mathematics)*. Cambridge University Press, July 1998.

[34] Okubo, A. Dynamical aspects of animal grouping: swarms, schools, flocks, and herds. In *Adv. Biophys* (1986), pp. 1–94.

[35] Parker, L. Alliance: An architecture for fault tolerant multi-robot cooperation. *IEEE Transactions on Robotics and Automation 14* (1998), 220–240.

[36] Pinciroli, C., Trianni, V., O'Grady, R., Pini, G., Brutschy, A., Brambilla, M., Mathews, N., Ferrante, E., Di Caro, G., Ducatelle, F., Birattari, M., Gambardella, L. M., and Dorigo, M. ARGoS: a modular, parallel, multi-engine simulator for multi-robot systems. *Swarm Intelligence 6*, 4 (2012). In press.

[37] Pollett, P. Integrals for continuous-time markov chains. *Mathematical Biosciences*, 182 (2003), 213–225.

[38] Pollett, P. K., and Stefanov, V. T. Path integrals for continuous-time markov chains. *J. Appl. Prob. 39* (2002), 901–904.

[39] Puri, S. A class of stochastic models of response afer infection in the absence of defense mechanism. In *Proc. Fifth Berkeley Symp. on Math. Statist. and Prob.* (1967), vol. 4, pp. 511–535.

[40] Scheidler, A. Dynamics of majority rule with differential latencies. *Physical Review E 83, 031116* (2011).

[41] Shaw, F., and Klavins, E. Distributed estimation and control for stochastically interacting robots. *Proceedings of the IEEE Conference on Decision and Control* (2009).

[42] Soysal, O., and Şahin, E. A macroscopic model for probabilistic aggregation in swarm robotic systems. In *Lecture Notes in Computer Science* (2006), pp. 27–42.

[43] Stefanov, V. Mean passage times for tridiagonal transition matrices. *J. Appl. Prob. 32* (1995), 84.

[44] Stefanov, V., and Wang, S. A note on integrals for birth-death processes. *Mathematical Biosciences 168* (2000), 161–5.

[45] Sugawara, K., Sano, M., Yoshihara, I., and Abe, K. Cooperative behavior of interacting robots. *Artificial Life and Robotics 2* (1998), 62–67.

[46] Syski, R. *Passage Times for Markov Chains.* IOS Press, Amsterdam, 1992.

[47] Winfield, A., and Nembrini, J. Safety in numbers: fault-tolerance in robot swarms. *International Journal of Modelling, Identification and Control* (2006).