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Dynamics of Majority Rule with Differential Latencies

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(Dated: February 23, 2011)

We investigate the dynamics of the majority-rule opinion formation model when voters experience *differential latencies*. With this extension, voters that just adopted an opinion go into a latent state during which they are excluded from the opinion formation process. The duration of the latent state depends on the opinion adopted by the voter. The net result is a bias towards consensus on the opinion that is associated with the shorter latency. We determine the exit probability and time to consensus for systems of N voters. Additionally, we derive an asymptotic characterisation of the time to consensus by means of a continuum model.

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I. INTRODUCTION

Binary-choice opinion formation models have recently received much attention from the statistical physics community [1]. They try to model consensus formation in populations of interacting voters. Typically, these models consist of N voters where, at any given time, each voter has one of the two possible opinions A or B. Voter's opinions are influenced by the opinions of other (neighbouring) voters by means of the repeated application of simple rules in the population. Prominent and well studied examples of binary-choice opinion formation models are the Voter model [12], the Sznajd model [14] or the Majority-Rule (MR) model [5]. The latter was originally proposed to capture the consensus formation in public debates and has been extensively studied in recent years (see, e.g., [2, 3, 6, 7, 9–11, 13]).

In the MR model the following two steps are applied. First, a group of random voters is selected and second, the voters in this group adopt the opinion that is favoured by the majority in the group. The repeated application of these two steps eventually drives the population to consensus, that is, a state in which all voters have the same opinion. The opinion on which consensus is reached is determined by the initial fractions of A and B voters. More precisely, the majority rule amplifies an existing opinion bias: with high probability the voters end up with the opinion that was initially in the majority.

Lambiotte et. al. [11] extend the MR model with the concept of latency: after voters have adopted an opinion they go temporarily into a *latent* state in which they cannot be influenced by other voters. However, they still participate in the opinion formation process and influence other voters. This extension leads to a rich dynamic behaviour that depends on the duration of the latent state.

Montes de Oca et. al. [4] introduce the concept of *differential latencies* in the MR model. Here the opinion adopted by a voter determines the duration the voter stays latent. In contrast to Lambiotte et. al.'s model

voters are excluded from the opinion formation if they are latent. As a consequence, voters that favour the opinion that is associated with the shorter latency participate more often in the application of the MR. This bias in the opinion formation process was shown to drive the voters with higher probability to consensus on the opinion that is associated with the shorter latency. Based on this finding Montes de Oca et. al. present a decentralized decision making method for groups of artificial agents. Here the term agent refers to an autonomously deciding and acting entity like, for example, a robot. Given two possible actions (opinions) that take different time to execute (latencies) the agents can collectively find the action which is associated with the faster execution (shorter latency). For example, it was shown that with the proposed method a group of robots is able to decide on the shorter of two paths between two locations without the need to measure travel times. The results presented in [4] are mainly obtained numerically and with normally distributed latencies.

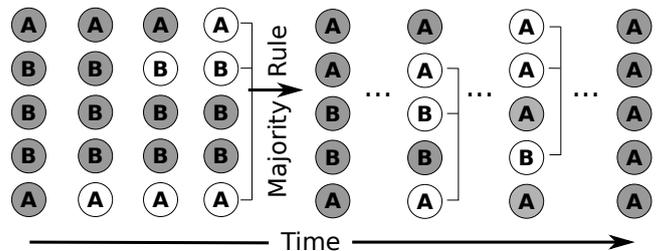


FIG. 1. Illustration of the update; gray voters are latent, white are non-latent; as soon as three voters became non-latent the MR is applied and the voters go back into latent state

The goal of this letter is to study the dynamics of the MR model with differential latencies analytically and to support the findings of [4] from a theoretical point of view. To this end, we investigate the following model (see also Figure 1): All voters start latent. The duration a voter stays latent follows an exponential distribution whose mean depends on the voter's opinion. Without loss of generality the mean time voters with opinion A stay

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latent is 1 and the mean duration of the latent state for voters with opinion B is $1/\lambda$ with $0 < \lambda \leq 1$. As soon as three voters have left the latent state the MR is applied and the voters go back into a latent state. Thus, never more than three voters are non-latent at any given time. Note that this simplifies the model presented in [4], where an arbitrary but fixed fraction of all voters stays non-latent.

II. EXIT PROBABILITY

In the following we estimate the exit probability E_n , that is, the probability that a system of N voters that starts with n voters for opinion A eventually finds consensus on A. Let n be the number of voters that currently vote for A and $x = n/N$ denote the density of A voters. The probability p that a voter that leaves the latent state has opinion A is given by

$$p = \frac{x}{x + \lambda(1-x)} \quad (1)$$

Note p only depends on x (we assume N to be large and neglect that p changes slightly when one or two voters already left the latent state).

E_n obeys the master equation:

$$E_n = w_+ E_{n+1} + w_- E_{n-1} + w_* E_n \quad (2)$$

with hopping probabilities :

$$\begin{aligned} w_+ &= 3p^2(1-p) \\ w_- &= 3p(1-p)^2 \\ w_* &= p^3 + (1-p)^3 \end{aligned}$$

We substitute these into (2), write $E_{n\pm 1} \rightarrow E(x \pm \delta x)$ and expand to second order in δx :

$$0 = 3p(1-p)(2p-1) \frac{\partial E}{\partial x} + \frac{1}{2} 3p(1-p) \delta x \frac{\partial^2 E}{\partial x^2} \quad (3)$$

Substituting (1) into (3) and letting $\delta x = \frac{1}{N}$ finally leads to

$$0 = 2N \left(\frac{2x}{x + \lambda(1-x)} - 1 \right) \frac{\partial E}{\partial x} + \frac{\partial^2 E}{\partial x^2} \quad (4)$$

The solution of this equation with respect to the boundary conditions $E(0) = 0$ and $E(1) = 1$ is

$$E(x) = \frac{I(x)}{I(1)} \quad (5)$$

where for the case that $\lambda = 1$

$$I(x) = \int_0^x e^{2N(y-y^2)} dy \quad (6)$$

and for $0 < \lambda < 1$

$$I(x) = \int_0^x e^{\frac{2Ny(\lambda+1)}{\lambda-1}} [y + \lambda(1-y)]^{\frac{4N\lambda}{(1-\lambda)^2}} dy \quad (7)$$

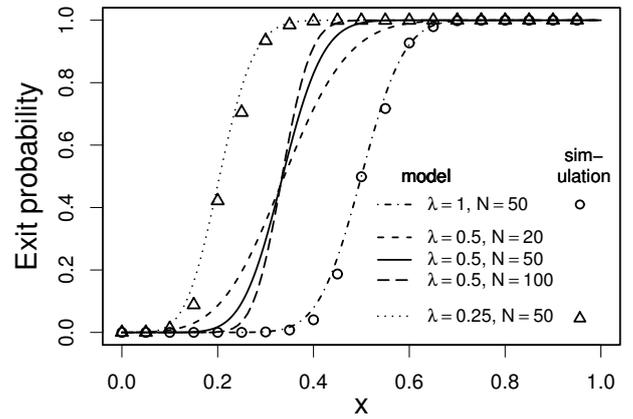


FIG. 2. Exit probability for 50 voters and different latencies; Comparison of the analytical model and simulation results

Figure 2 depicts $E(x)$ and results of Monte-Carlo simulations of 50 voters for latencies $\lambda \in \{1, 0.5, 0.25\}$ (Note that all presented simulation results are averaged over 1000 independent runs). Clearly, for equal latencies ($\lambda = 1$) the model is equivalent to the MR model. In this case the density $x = 0.5$ marks the *critical (initial) density* of A voters that determines the consensus state: systems that initially start with $x < 0.5$ tend to find consensus on B, whereas systems that start with $x > 0.5$ find consensus on A with high probability.

The results for $\lambda \neq 1$ show that differential latencies influence the exit probability significantly. The more the latencies for the two opinions differ (the smaller λ) the more is the critical initial density shifted towards smaller values. More precisely, the critical density is now given by $x = \lambda/(1 + \lambda)$. This value corresponds to a system state in which the voters for A and B leave the latent state with the same rate.

In the standard MR model the exit probability converges to a step function for growing N . This is still valid for the MR model with differential latencies (see the results for $\lambda = 0.5$ given in Figure 2). Clearly, for very large N , Formula (4) is mainly determined by the drift term and only near the critical density the drift term becomes comparable to the diffusion term.

III. TIME TO CONSENSUS

A. Master Equation

The expected time T_n to reach consensus from a state where n voters have opinion A obeys the master equation

$$T_n = \delta t + w_+ T_{n+1} + w_- T_{n-1} + w_* T_n, \quad (8)$$

where δt denotes the expected time between two applications of the MR. This value is not constant but depends on the actual fractions of opinions in the system.

To determine δt we use the fact that the minimum of exponentially distributed random variables is exponentially distributed with parameter equal to the sum of the parameters of the single distributions. Thus, the expected time between two voters leaving latent state is distributed exponentially with parameter $\mu = n + \lambda(N - n)$. We can hence estimate the time between two applications of the MR as $\delta t = 3/\mu$ (Note that this is only valid for large N because it does not take the voters into account that might have already left the latent state). Inserting δt in (8) and expanding to second order results in the equation

$$Nx(1-p)(2p-1)\frac{\partial T}{\partial x} + \frac{1}{2}x(1-p)\frac{\partial^2 T}{\partial x^2} = -1. \quad (9)$$

Figure 3 shows the numerical solution of (9) for $N = 50$ and boundary conditions $T(0) = 0$ and $T(1) = 0$ together with results obtained in a Monte Carlo simulation. Without differential latencies ($\lambda = 1$) the results of the exact solution given in [9] are resembled. As expected, these values are symmetrical to $x = 0.5$ (no initial bias).

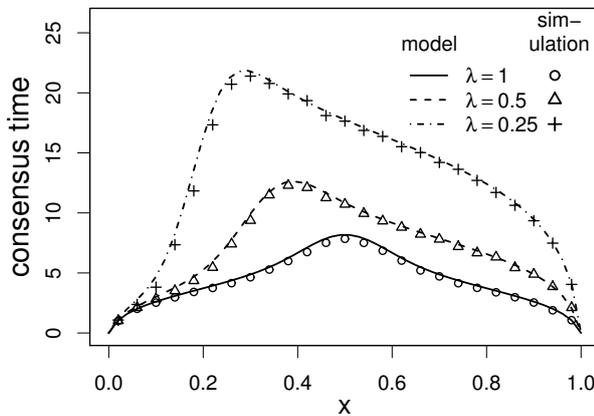


FIG. 3. Consensus time T_n^N versus n (initial number of voters for A) for $N = 50$ voters; comparison of the model predictions with results gained in a Monte-Carlo simulation

Differential latencies ($\lambda \neq 1$) increase the time to find consensus. This is because on average less updates per unit of time are applied. Moreover, caused by the shift of the critical densities the curves are not symmetrical. Although opinion A is associated with the shorter latency, the time for a system biased to A takes longer to converge compared to a system equally biased to B (compare, e.g., $T(0.1)$ and $T(0.9)$). The reason is that the rate of change mainly depends on the rate voters that are in the minority leave the latent state. For example, consider a state which is biased to A. If B voters become non-latent there is thus a high probability that they will be “convinced”. However, this happens only with rate λ . On the other hand, if the system is biased to B the rate at which A voters are convinced is 1 and thus the voters will find consensus faster in this case.

B. Continuum Model Approximation

In the following we characterise the asymptotic behaviour of the time to reach consensus. The formal solution of the Fokker-Planck equation (9) with respect to given boundary conditions can be derived (for an ansatz see, e.g., [8]). However, this solution is complex and hard to analyse. We therefore choose to approximate the consensus time by means of a continuum model. This can easily be done and we show (experimentally) that this approximation becomes more accurate for large N .

In an unit time step the overall fraction of voters that become non-latent is $x + \lambda(1 - x)$. The probability that in a triple of these voters at least two voters have opinion A is given by $3p^2(1 - p) + p^3$. This leads to the model:

$$\dot{x} = -x + [3p^2 - 2p^3](x + \lambda(1 - x)) \quad (10)$$

Figure 4 depicts \dot{x} for $\lambda \in \{1, 0.5, 0.25\}$. The zeros of \dot{x} , i.e., the stationary solutions of (10) are the (stable) consensus states $[x = 0]$ and $[x = 1]$ and the (unstable) equilibrium point $[x = \lambda/(1 + \lambda)]$. The latter marks the critical density that separates the flow to the consensus states.

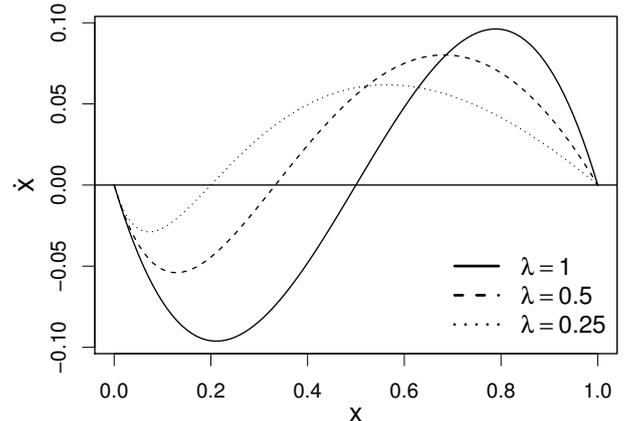


FIG. 4. Change of the number of voters favouring opinion A dependent on the actual fraction of voters with this opinion

To estimate the time until consensus for a finite number of voters we rewrite equation (10) in the partial fraction expansion and integrate over a suitable chosen interval. More precisely, we integrate from point $a_0 = n/N$ to a point a_∞ sufficiently near the respective consensus state. The point a_∞ corresponds to a state in which the system deviates from consensus only in a single voter. Hence, we thereby get an estimation of the time it takes the system to reach a state where only one last application of the majority is needed to reach consensus. Recall that for any state $a_0 > \lambda/(1 + \lambda)$ greater than the critical density the system finds consensus on A, whereas for $a_0 < \lambda/(1 + \lambda)$ the system will develop consensus on B. Thus, the time to reach consensus in a system of N voters that starts with n voters for opinion A can be

approximated as

$$T_n^N \approx \int_{\frac{n}{N}}^{a_\infty} \left[\frac{4}{a(\lambda+1) - \lambda} - \frac{1}{(a-1)\lambda} - \frac{1}{a} \right] da \quad (11)$$

with

$$a_\infty = \begin{cases} 0 + \frac{1}{N} & \text{if } \frac{n}{N} < \frac{\lambda}{1+\lambda}, \\ 1 - \frac{1}{N} & \text{if } \frac{n}{N} > \frac{\lambda}{1+\lambda}. \end{cases}$$

Figure 5 depicts results of this approximation together with solutions of the model (9). For few voters ($N = 50$) the models differ the most near the critical density. This is because systems that are initially biased towards one opinion still can reach consensus on the other opinion. This fact is taken into account by the model obtained from the master equation. The approximation model, on the other hand, assumes that the critical density determines the fate of the system. However, for larger N the approximation becomes more accurate (compare results for $N = 10.000$ in Figure 5).

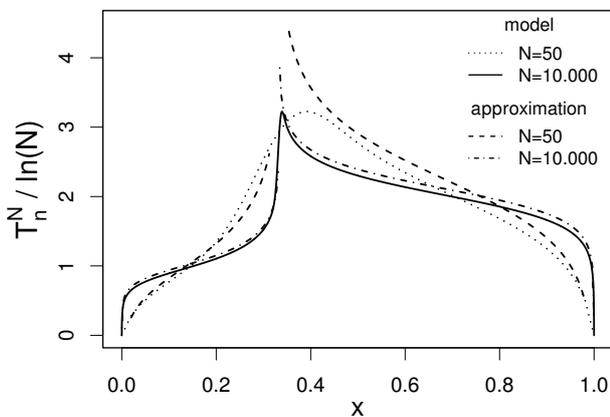


FIG. 5. Consensus time T_n^N versus $x = n/N$ (initial fraction of voters for opinion A) for different number of voters

In the following we determine T_{max} , the maximal time until consensus is reached for a given number of voters N . To estimate this time we integrate from a point that deviates only in one voter from the critical initial density:

$$T_{max} \approx T_{\lfloor \frac{\lambda}{1+\lambda} N + 1 \rfloor}^N \sim \frac{5\lambda + 1}{\lambda(1 + \lambda)} \ln N \quad (12)$$

For $\lambda = 1$ and large N this result reduces to the asymptotic behaviour $T_{max} \sim 3 \ln N$ that was also derived in [2]

for the standard MR model. Moreover, as long as the latencies for the two opinions are comparable the maximal consensus time grows asymptotically as $T_{max} \sim \ln N$. However, if only very few voters for B go into non-latent state the consensus time is mainly determined by this flow rate. This is reflected by the fact that for very long latencies $\lambda \ll 1$ the consensus time grows as $T_{max} \sim 1/\lambda$.

If we consider densities sufficiently far from the critical density, that is, if $x - \lambda/(1 + \lambda)$ becomes comparable to x the consensus time T_n^N drops quickly (see Figure 5). For the MR model without latencies such a change in the amplitude in the consensus time to $T_n^N \sim \ln N$ was also mentioned in [2] and in [9]. However, in the case of differential latencies the drop of the amplitude is not symmetrical. As already explained, at a certain point in the evolution of the system the time until consensus is mainly determined by the rate the voters that are in the minority leave the latent state. This is the reason why for $x < \lambda/(1 + \lambda)$ the consensus time drops to $T_n^N \sim \ln N$, but for $x > \lambda/(1 + \lambda)$ it drops to $T_n^N \sim 1/\lambda \ln N$.

IV. CONCLUSIONS

The introduction of differential latencies in the MR model leads to a bias towards consensus on the opinion that is associated with the shorter latency. This effect increases with the number of voters N and with the ratio between the latency times λ . Moreover, the maximal time to find consensus scales as $1/\lambda \ln N$.

These results particularly apply for systems of voters that start unbiased (i.e., with initial density $x = 0.5$). Hence, from the point of view of the application in a decision making method [4], this confirms two important scalability results. First, the decision method improves when the number of agents is increased and second, since it scales with $\ln N$, the time needed to find a decision is only marginally influenced by an increased number of agents.

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