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Dynamics of Majority Rule with Differential Latencies

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We investigate the dynamics of the majority-rule opinion formation model when voters experience *differential latencies*. With this extension, voters that just adopted an opinion go into a latent state during which they are excluded from the opinion formation process. The duration of the latent state depends on the opinion adopted by the voter. The net result is a bias towards consensus on the opinion that is associated with the shorter latency. We determine the exit probability for systems of N voters. Additionally, we estimate the time needed to reach consensus by means of a continuum model. Finally, we briefly describe an application of the proposed model in computer science.

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Binary-choice opinion formation models have recently received much attention from the statistical physics community [1]. They try to model consensus formation in populations of interacting voters. Typically, these models consist of N voters. At any given time, each voter has one of the two possible opinions A or B. Each voter's opinion is influenced by the opinions of other voters. The system evolves by means of the repeated application of simple rules in the population. In the *majority-rule* opinion formation model this rule consists in selecting a group of random voters and let them adopt the opinion of the majority in this group. The repeated application of this rule eventually drives the population to consensus on one of the two opinions. The opinion on which consensus is reached is determined by the initial fractions of A and B voters. More precisely, the majority rule tends to amplify an existing opinion bias: with high probability the voters will all end up with the opinion that was initially in the majority.

The majority-rule opinion dynamics model was originally proposed to capture the consensus formation in public debates [5] and has been extensively studied in recent years (see, e.g., [2, 3, 6, 8]). Lambiotte et. al. [7] extend the majority-rule model with the concept of latency: after voters have adopted an opinion they go temporarily into a *latent* state in which they cannot be influenced by other voters. However, they can still participate in the opinion formation process and influence other voters. It was shown that this results in a rich dynamic behavior that depends on the duration of the latency period.

Montes de Oca et. al. [4] introduce the concept of *differential latencies* in the majority-rule opinion dynamics model. With this extension the opinion adopted by a voter determines the duration the voter stays latent. In contrast to Lambiotte et. al. 's model voters are excluded from the opinion formation if they are latent. As a consequence, voters that favor the opinion that is associated with the shorter latency participate more often in the application of the majority rule. This bias in the opinion formation process was shown to drive the voters with higher probability to consensus on the opinion that is associated with the shorter latency. The results presented in [4] are mainly obtained numerically and with normally distributed latencies.

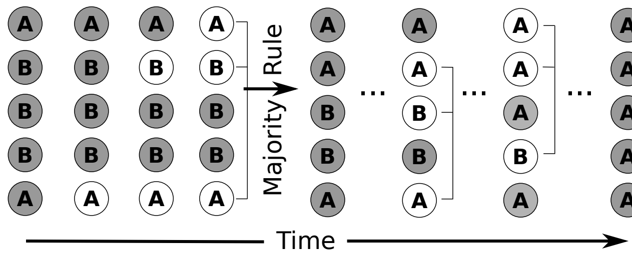

FIG. 1. Illustration of the update; gray voters are latent, white are non-latent; as soon as three voters became non-latent the majority rule is applied and the voters go back into latent state

In this letter we study the dynamics of the majority-rule with differential, exponentially distributed latencies. Figure 1 depicts how the update in the investigated model applies. All voters start latent. The durations the voters stay latent follow exponential distributions whose means depend on the voters opinions. Without loss of generality the mean time voters with opinion A stay latent is 1 and the mean duration of the latent state for voters with opinion B is $1/\lambda$ with $0 < \lambda \leq 1$. As soon as three voters have left the latent state the majority rule is applied and the voters go back into a latent state. Thus, never more than three voters are non-latent at any given time. Note that this simplifies the model presented in [4], where an arbitrary but fixed fraction of all voters stays non-latent.

Exit Probability

In the following we estimate the exit probability E_n, that is, the probability that a system of N voters that

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starts with n voters for opinion A eventually finds consensus on A. Let n be the number of voters that currently vote for A. Let $x = n/N$ denote the density of A voters. The probability p that a voter that leaves the latent state has opinion A is given by

$$p = \frac{x}{x + \lambda(1-x)} \tag{1}$$

Note that N is assumed to be large. Therefore this estimation of p solely dependents on x and we neglect the fact that p also depends on the voters that might have already left the latent state.

E_n obeys the master equation:

$$E_n = w_+ E_{n+1} + w_- E_{n-1} + w_* E_n \tag{2}$$

with hopping probabilities :

$$\begin{aligned} w_+ &= 3p^2(1-p) \\ w_- &= 3p(1-p)^2 \\ w_* &= p^3 + (1-p)^3 \end{aligned}$$

We substitute these into (2), write $E_{n\pm1} \to E(x \pm \delta x)$ and expand to second order in δx:

$$0 = 3p(1-p)(1-2p)\frac{\partial E}{\partial x} + \frac{1}{2}3p(1-p)\delta x \frac{\partial^2 E}{\partial x^2} \tag{3}$$

Substituting (1) into (3) and letting $\delta x = \frac{1}{N}$ finally leads to

$$0 = 2N\left(1 - \frac{2x}{x + \lambda(1-x)}\right)\frac{\partial E}{\partial x} + \frac{\partial^2 E}{\partial x^2} \tag{4}$$

The solution of this equation with respect to the boundary conditions $E(0) = 0$ and $E(1) = 1$ is

$$E(x) = \frac{I(x)}{I(1)} \tag{5}$$

where for the case that $\lambda = 1$

$$I(x) = \int_0^x e^{2N(y-y^2)} \mathrm{d}y \tag{6}$$

and for $0 < \lambda < 1$

$$I(x) = \int_0^x e^{\frac{2Ny(\lambda+1)}{\lambda-1}} [y + \lambda(1-y)]^{\frac{4N\lambda}{(1-\lambda)^2}} \mathrm{d}y \tag{7}$$

Figure 2 depicts this analytical result in comparison to data obtained in a Monte Carlo simulation of 50 voters for different latency values λ. If the opinions are associated with equal latencies ($\lambda = 1$) the model is equivalent to the majority rule. The density $x = 0.5$ marks the *critical (initial) density* of A voters that determines the final consensus state: systems that initially start with $x < 0.5$ tend to find consensus on B, whereas systems that start with $x > 0.5$ find consensus on A with high probability.

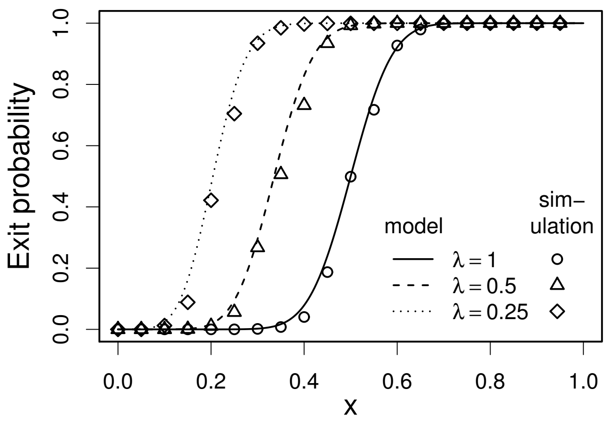

FIG. 2. Exit probability for 50 voters and different latencies; Comparison of the analytical model and simulation results

The results for $\lambda \neq 1$ show that differential latencies influence the exit probability significantly. Depending on λ the critical initial density is shifted towards smaller values. More precisely, the critical density is now determined by $x = \lambda/(1+\lambda)$. This value corresponds to a system state in which the voters for A and B leave the latent state in the same rates.

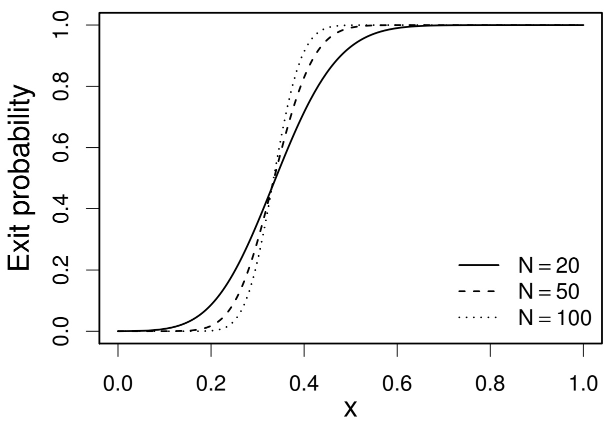

FIG. 3. Exit probability for different numbers of voters if opinion B is associated with a latency duration twice as long as as the latency associated with opinion A ($\lambda = 0.5$)

Figure 3 shows the influence of the number of voters on the exit probability for the case of differential latencies ($\lambda = 0.5$). As in the majority-rule model without differential latencies the exit probability converges to a step function the more voters are present. Clearly, for very large N, Formula 4 is mainly determined by the drift term and only near the critical density the drift term becomes comparable to the diffusion term.

Time to Consensus

In the following we propose a continuum model of the majority-rule with differential latencies and use it to estimate the time the voters need to find consensus. Within a unit time step the overall fraction of voters that become non-latent is

$$r = x + \lambda(1 - x).$$

The probability that in a triple of these voters at least two voters have opinion A is given by $3p^2(1 - p) + p^3$. This leads to the model:

$$\dot{x} = -x + r\left[3p^2(1 - p) + p^3\right] \quad (8)$$

Figure 4 depicts $\dot{x}$ for $\lambda \in \{1, 0.5, 0.25\}$. The zeros of $\dot{x}$, that is, the stationary solutions of (8) are the (stable) consensus states $[x = 0]$ and $[x = 1]$ and the (unstable) equilibrium point $[x = \lambda/(1 + \lambda)]$. The latter marks the critical initial density that separates the flow to the two consensus states.

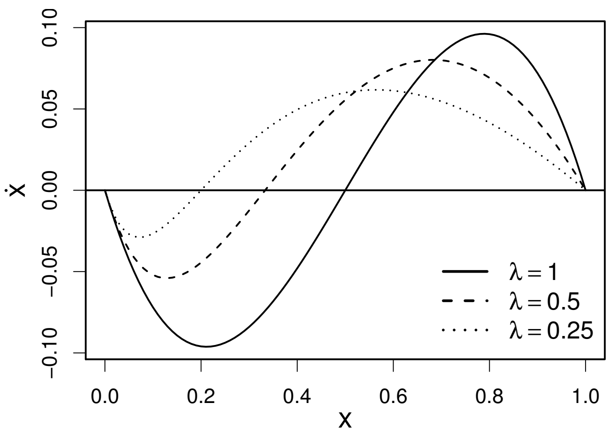

FIG. 4. Change of the number of voters favouring opinion A dependent on the actual fraction of voters with this opinion

To estimate consensus time for a finite number of voters we rewrite equation (8) in the partial fraction expansion and integrate it with suitable chosen initial and final states. More precisely, we ask how long it takes in the continuum model to evolve from point $a_0 = n/N$ to a point a_∞ sufficiently near to the respective consensus state. For any state $a_0 > \lambda/(1 + \lambda)$ greater than the critical density the system finds consensus on A, whereas for $a_0 < \lambda/(1 + \lambda)$ the system will develop consensus on B. Thus, the time to reach consensus in a system of N voters that starts with n voters for opinion A can be approximated as

$$T_n^N \approx \int_{\frac{n}{N}}^{a_\infty} \left[\frac{4}{a(\lambda + 1) - \lambda} - \frac{1}{(a - 1)\lambda} - \frac{1}{a}\right] da \quad (9)$$

with

$$a_\infty = \begin{cases} 0 + \frac{1}{N} & \text{if } \frac{n}{N} < \frac{\lambda}{1+\lambda}, \\ 1 - \frac{1}{N} & \text{if } \frac{n}{N} > \frac{\lambda}{1+\lambda}. \end{cases}$$

Figure 5 shows a comparison between this estimation and results obtained in a Monte Carlo simulation with $N = 50$ voters. First consider the results for $\lambda = 1$, that is, the case without differential latency. It can be seen that the theoretical predictions overestimate the simulation results slightly (additional experiments have shown that the estimation gets better with more voters). However, the shape of curve resembles the results of the exact solution given in [6]. As expected, the curves are symmetrical to $n = 25$ (no initial bias).

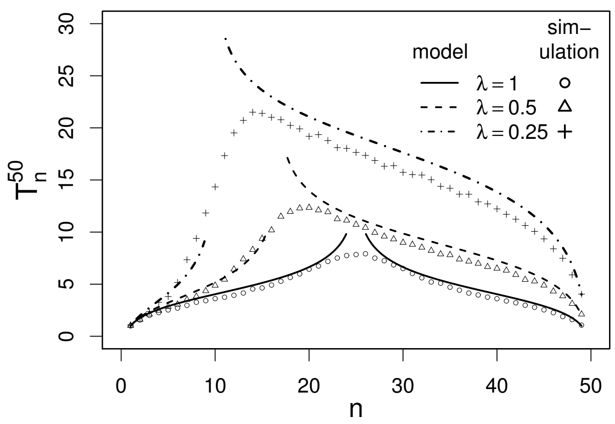

FIG. 5. Consensus time T_n^N versus n (initial number of voters for A) for $N = 50$ voters; comparison of the model predictions with results gained in a Monte-Carlo simulation

It is obvious that differential latencies $\lambda \neq 1$ must increase the time the voters need to converge. The reason is that on average less updates per of unit time are applied. Moreover, caused by the shift of the critical density values the curves are not symmetrical since it still takes the longest time to find consensus if the system starts near the shifted critical densities. The results of the simulation and the model estimation differ the most near the critical densities. The reason is that in the Monte Carlo simulation systems that are initially biased towards one opinion still can reach consensus on the other opinion, whereas the theoretical estimation assumes that the critical density determines the fate of the system.

Although opinion A is associated with the shorter latency, the time for a system biased to A takes longer to converge compared to a system equally biased to B . The reason is that the rate of change depends mainly on the rate voters that are in the minority leave the latent state.

In the following we determine T_{max}, the maximal time until consensus is reached for a given number of voters N. To estimate this time we integrate from a point that deviates only in one voter from the critical initial density:

$$T_{max} \approx T^N_{\left[\frac{\lambda}{1+\lambda} N + 1\right]} \sim \frac{5\lambda + 1}{\lambda(1 + \lambda)} \ln N \quad (10)$$

For $\lambda = 1$ and large N this result reduces to the asymptotic behavior $T_{max} \sim 3 \ln N$ derived in [2] from a continuum model for the majority rule without differential latencies. Moreover, as long as the latencies for the two opinions are comparable the maximal consensus time grows asymptotically as $T_{max} \sim \ln N$. However, if only very few voters for B go into non-latent state the consensus time is mainly determined by this flow rate. This is reflected by the fact that for very long latencies $\lambda \ll 1$ the consensus time grows as $T_{max} \sim 1/\lambda$.

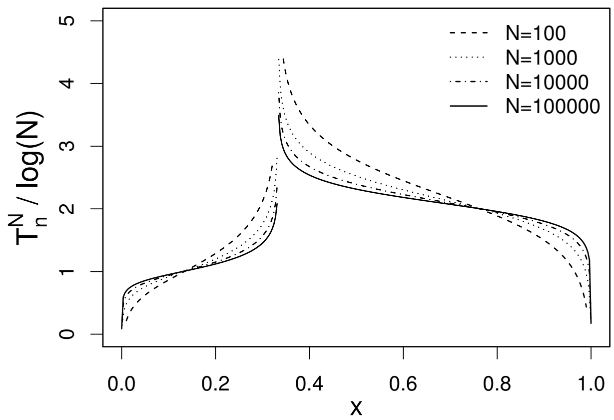

FIG. 6. Consensus time T_n^N versus $x = n/N$ (initial fraction of voters for opinion A) for different number of voters

If we consider densities sufficiently far from the critical density, that is, if $x - \lambda/(1+\lambda)$ becomes comparable to either 0 or x the consensus time T_n^N drops quickly (see Figure 6). For the majority-rule without latencies such a change in the amplitude in the consensus time to $T_n^N \sim \ln N$ was also mentioned in [2] and in [6]. However, in the case of differential latencies the drop of the amplitude is not symmetrical. For $x < \lambda/(1+\lambda)$ it still drops to $T_n^N \sim \ln N$. But for $x > \lambda/(1+\lambda)$ the consensus time drops to $T_n^N \sim 1/\lambda \ln N$.

Decentralized Decision Making

As shown the majority-rule with differential latencies leads to interesting dynamics. For equal latencies the standard majority-rule model is resembled. If the latencies for the two opinions differ the critical initial density that determines the fate of the system is shifted in the direction of the opinion that is associated with the shorter latency.

Particularly unbiased systems, i.e., systems that start with equal proportions of voters for A and B, tend to find consensus on the opinion that is accosiated with the shorter latency. Based on this finding Montes de Oca et. al. [4] present a decentralized decision making method for groups of artificial agents. Here the term agent refers to an autonomously deciding and acting entity like, for example, a robot. In large groups of such agents typically only local interactions occur and no agent can possess global knowledge. Thus, to make global (group wide) decisions there is the need for decentralized methods in which such decisions emerge from local interactions of the agents. The approach presented provides such a method: given two possible actions (opinions) that take different time to execute (latencies) the agents can collectively find the action which is associated with the faster execution (shorter latency). Such a method is very interesting because typically short execution times lead to better system performance. For example, it was shown that with the proposed method a group of robots is able to decide on the shorter of two paths between two locations.

With respect to the experimental results given in [4] this paper verifies the following observations. Groups of agents that start without a-priori knowledge (i.e., unbiased $n = N/2$) tend to find consensus on the action associated with the shorter action execution time. The quality of the reached decision (its correctness) increases with the number of agents and with the difference in the action execution times. Moreover, for "reasonable" action execution times the time to find a decision scales as $\ln N$. This is an important result since it shows that the time the agents need to come up with a decision scales very well with their number.

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