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# On the Invariance of Ant Colony Optimization

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## Abstract

*Ant colony optimization* (ACO) is nowadays one of the most promising metaheuristics, and an increasing amount of research has been devoted to its empirical and theoretical analysis. Some authors believe that the performance of ant colony optimization depends somehow on the scale of the problem instance under analysis. The issue has been recently raised explicitly [1] and the *hyper-cube framework* has been proposed to handle this supposed dependency of ACO on the scale of the instances.

This paper shows that the ACO internal state—commonly referred to as the *pheromone* in the literature—indeed depends on the scale of the problem at hand. Nonetheless, we formally prove that this does not affect the external behavior of the algorithm. In other words, the sequence of solutions produced by ACO does not depend on the scale of the problem instance under analysis.

Moreover, the paper introduces three variations of the three most widely adopted algorithms belonging to the ant colony optimization family. We formally show that the algorithms we propose are *functionally equivalent* to the original ones, that is, for any given instance, these algorithms produce the same sequence of solutions as the original ones. Nonetheless, in these new algorithms, also the internal state is independent of the scale of the problem instance at hand.

## 1 Introduction

Ant colony optimization (ACO) [2] is a successful metaheuristics inspired by the foraging behavior of ants [3]: In order to find the shortest path from a nest to a food source, ant colonies exploit a positive feedback mechanism by using a form of indirect communication called stigmergy [4], based on the laying and detection of pheromone trails. In ant colony optimization, a generic combinatorial optimization problem at hand is encoded into a constrained shortest path problem. The search is an iterative process: A number of paths are generated in a Monte Carlo fashion on the basis of a probabilistic model whose parameters are called *artificial pheromone*—or more simply *pheromone*. The cost of these paths is used to modify the pheromone and therefore to bias the generation of further paths towards promising regions of the search space [5].

The ant colony optimization framework has been explicitly defined by Dorigo et al. in 1999 [6], and comprises a number of algorithms including the original ant system [7, 8, 9], ant colony system [10], and *MAX-MIN* ant system [11, 12]. A vast literature exists on ant colony optimization and on its application to a large number of problems. We refer the reader to Dorigo and Stützle [2] for a comprehensive account.

Recently, the *hyper-cube framework* [1] has been introduced with the aim of implementing ant colony optimization algorithms that are invariant with respect to a linear rescaling of problem instances. The need for the introduction of the hyper-cube framework has been explicitly motivated by the observation that

*in standard ACO algorithms the pheromone values and therefore the performance of the algorithms, strongly depend on the scale of the problem.* [1]

In this paper, we formally show that this statement is only partially correct: Indeed, in standard ant colony optimization algorithms the pheromone values (and the heuristic information) depend on the scale of the problem. Nonetheless, the sequence of solutions they find is independent of the scaling.

For definiteness, the paper focuses on ant system,  $\mathcal{MAX}\text{-}\mathcal{MIN}$  ant system, and ant colony system, which are the three most representative algorithms in the ant colony optimization family. Further, we focus on the traveling salesman problem, which is the problem to which ant colony optimization was first applied. Moreover, the traveling salesman problem is widely known and, given its simplicity, is particularly suitable for this kind of analysis. The theorems we enunciate are proved for this specific problem and then the conditions under which these results extend to other problems are briefly discussed.

As a second main contribution, the paper proposes variants of ant system,  $\mathcal{MAX}\text{-}\mathcal{MIN}$  ant system, and ant colony system. These new algorithms, called *strongly-invariant ant system* (siAS), *strongly-invariant  $\mathcal{MAX}\text{-}\mathcal{MIN}$  ant system* (siMMAS), and *strongly-invariant ant colony system* (siACS), are equivalent to their original counterparts but, beside being able to find the same solutions irrespectively of the scaling, also have the property that the pheromone and the heuristic values do not depend on the scaling itself. This property is desirable for at least two reasons: On the one hand, it reduces numerical problems and makes the algorithm more robust; on the other hand, it improves the readability of the solution process and makes the analysis of ACO algorithms easier.

The rest of the paper is organized as follows. Section 2 introduces some preliminary concepts. Sections 3, 4, and 5 deal with ant system,  $\mathcal{MAX}\text{-}\mathcal{MIN}$  ant system, and ant colony system, respectively. In these sections the three algorithms are formally defined and then it is formally proved that the sequence of solutions they produce does not depend on the scale of the problem instance under analysis. Moreover, these sections propose the *strongly-invariant* versions of the three algorithms and formally study their properties. As we will detail in the following, in order to be invariant to the scale of the problem, siAS, siMMAS, and siACS need to perform some extra processing and this could, in principle, slow down significantly the computation. Section 6 proposes an empirical analysis that shows that the difference in speed between the strongly-invariant versions and the corresponding classical ones is negligible. Section 7 concludes the paper.

## 2 Preliminary definitions

This section introduces a number of fundamental concepts that will be needed in the following.

**Definition 1 (Linear transformation of a traveling salesman instance).** With  $\bar{I} = fI$ ,  $f > 0$ , we indicate that the instance  $\bar{I}$  is a *linear transformation* of the instance  $I$ : The two instances have the same number of cities and the cost  $\bar{c}_{ij}$  of traveling from city  $i$  to city  $j$  in  $\bar{I}$  is  $f$  times the corresponding cost  $c_{ij}$  in instance  $I$ . Formally:

$$\bar{c}_{ij} = fc_{ij}, \text{ for all } \langle i, j \rangle. \quad (1)$$

*Remark 1.* The cost  $\bar{C}$  of a solution  $\bar{T}$  of instance  $\bar{I}$  is  $f$  times the cost  $C$  of the corresponding solution  $T$  of instance  $I$ . Formally:

$$(\bar{I} = fI) \wedge (\bar{T} = T) \implies \bar{C} = fC. \quad (2)$$

*Remark 2.* In the following, if  $x$  is a generic quantity that refers to an instance  $I$ , then  $\bar{x}$  is the corresponding quantity for what concerns instance  $\bar{I}$ , when  $\bar{I}$  is a linear transformation of  $I$ .

Ant colony optimization algorithms are *stochastic*: Solutions are constructed incrementally on the basis of stochastic decisions that are biased by the pheromone and by some heuristic information. The following hypothesis will be used in the paper.

**Hypothesis 1 (Pseudo-random number generator).** When solving two instances  $I$  and  $\bar{I}$ , the stochastic decisions taken while constructing solutions are made on the basis of random experiments based on pseudo-random numbers produced by the same pseudo-random number generator. We assume that this generator is initialized in the same way (for example, with the same seed) when solving the two instances so that the two sequences of pseudo-random numbers that are generated are the same in the two cases.

Similarly, when two algorithms  $A$  and  $\tilde{A}$  solve a same instance  $I$ , we assume that the pseudo-random number generators adopted by the two algorithms are the same and are initialized in the same way.

**Definition 2 (Invariance).** An algorithm  $A$  is **invariant** to linear transformations if the sequence of solutions  $S_I$  generated when solving an instance  $I$  and the sequence of solutions  $S_{\bar{I}}$  generated when solving an instance  $\bar{I}$  are the same, whenever  $\bar{I}$  is a linear transformation of  $I$ .

If  $A$  is a stochastic algorithm, it is said to be invariant if it is so under Hypothesis 1.

**Definition 3 (Strong and weak invariance).** An algorithm  $A$  is said to be **strongly-invariant** if, beside generating the same solutions on any two linearly related instances  $I$  and  $\bar{I}$ , it also enjoys the property that the heuristic information and the pheromone at each iteration are the same when solving  $I$  and  $\bar{I}$ . Conversely, the algorithm  $A$  is **weakly-invariant** if it obtains the same solutions on linearly related instances but the heuristic information and the pheromone assume different values.

If  $A$  is stochastic, it is said to be strongly-invariant (or weakly-invariant) if it is so under Hypothesis 1.

**Definition 4 (Functional equivalence).** Two algorithms  $A$  and  $\tilde{A}$  are *functionally equivalent*, or simply *equivalent*, if for any instance  $I$ , the sequence of solutions  $S_I$  generated by  $A$  and the sequence of solutions  $\tilde{S}_I$  generated by  $\tilde{A}$  are the same.

If  $A$  and  $\tilde{A}$  are stochastic, they are said to be equivalent if they are so under Hypothesis 1.

### 3 Ant system

Ant system is the original ant colony optimization algorithm proposed by Dorigo et al. [7, 8, 9].

**Definition 5 (Random proportional rule).** At the generic iteration  $h$ , suppose that ant  $k$  is in node  $i$ . Let  $\mathcal{N}_i^k$  be the set of feasible nodes. The node  $j \in \mathcal{N}_i^k$ , to which ant  $k$  moves, is selected with probability:

$$p_{ij,h}^k = \frac{[\tau_{ij,h}]^\alpha [\eta_{ij}]^\beta}{\sum_{l \in \mathcal{N}_i^k} [\tau_{il,h}]^\alpha [\eta_{il}]^\beta},$$

where  $\alpha$  and  $\beta$  are parameters,  $\tau_{ij,h}$  is the pheromone value associated with arc  $\langle i, j \rangle$  at iteration  $h$ , and  $\eta_{ij}$  represents *heuristic information* on the desirability of visiting node  $j$  after node  $i$ .

**Definition 6 (Heuristic information).** When solving the traveling salesman problem, the heuristic information  $\eta_{ij}$  is the inverse of the cost of traveling from city  $i$  to city  $j$ :

$$\eta_{ij} = \frac{1}{c_{ij}}, \text{ for all } \langle i, j \rangle.$$

**Definition 7 (Pheromone update rule).** At the generic iteration  $h$ , suppose that  $m$  ants have generated the solutions  $T_h^1, T_h^2, \dots, T_h^m$  of cost  $C_h^1, C_h^2, \dots, C_h^m$ , respectively. The pheromone on each arc  $\langle i, j \rangle$  is updated according to the following rule:

$$\tau_{ij,h+1} = (1 - \rho)\tau_{ij,h} + \sum_{k=1}^m \Delta_{ij,h}^k,$$

where  $\rho$  is a parameter called *evaporation rate* and

$$\Delta_{ij,h}^k = \begin{cases} 1/C_h^k, & \text{if } \langle i, j \rangle \in T_h^k; \\ 0, & \text{otherwise.} \end{cases} \quad (3)$$

**Definition 8 (Ant system).** Ant system is an ant colony optimization algorithm in which solutions are constructed according to the random proportional rule given in Definition 5, and the pheromone is updated according to the rule given in Definition 7. The evaporation rate  $\rho$ , the number of ants  $m$ , and the exponents  $\alpha$  and  $\beta$  are parameters of the algorithm.

When ant system is used for solving the traveling salesman problem, it is customary to initialize the pheromone as follows.

**Definition 9 (Nearest-neighbor pheromone initialization).** At the first iteration  $h = 1$ , the pheromone on all arcs is initialized to the value:

$$\tau_{ij,1} = \tau_{init} = \frac{m}{C^{nn}}, \text{ for all } \langle i, j \rangle,$$

where  $m$  is the number of ants considered at each iteration, and  $C^{nn}$  is the cost of the solution  $T^{nn}$  obtained by the nearest-neighbor heuristic.

The following theorem holds true.

**Lemma 1.** *The random proportional rule is invariant to concurrent linear transformations of the pheromone and of the heuristic information. Formally, for any two positive constants  $g_1$  and  $g_2$ ,*

$$(\bar{\tau}_{ij,h} = g_1\tau_{ij,h}) \wedge (\bar{\eta}_{ij} = g_2\eta_{ij}), \text{ for all } \langle i, j \rangle \implies \bar{p}_{ij,h}^k = p_{ij,h}^k, \text{ for all } \langle i, j \rangle,$$

where  $\bar{p}_{ij,h}^k$  is obtained on the basis of  $\bar{\tau}_{ij,h}$  and  $\bar{\eta}_{ij}$ , according to Definition 5.

*Proof.* According to Definition 5:

$$\begin{aligned} \bar{p}_{ij,h}^k &= \frac{[\bar{\tau}_{ij,h}]^\alpha [\bar{\eta}_{ij}]^\beta}{\sum_{l \in \mathcal{N}_i^k} [\bar{\tau}_{il,h}]^\alpha [\bar{\eta}_{il}]^\beta} = \frac{[g_1\tau_{ij,h}]^\alpha [g_2\eta_{ij}]^\beta}{\sum_{l \in \mathcal{N}_i^k} [g_1\tau_{il,h}]^\alpha [g_2\eta_{il}]^\beta} = \frac{[g_1]^\alpha [g_2]^\beta [\tau_{ij,h}]^\alpha [\eta_{ij}]^\beta}{\sum_{l \in \mathcal{N}_i^k} [g_1]^\alpha [g_2]^\beta [\tau_{il,h}]^\alpha [\eta_{il}]^\beta} \\ &= \frac{[g_1]^\alpha [g_2]^\beta [\tau_{ij,h}]^\alpha [\eta_{ij}]^\beta}{[g_1]^\alpha [g_2]^\beta \sum_{l \in \mathcal{N}_i^k} [\tau_{il,h}]^\alpha [\eta_{il}]^\beta} = \frac{[\tau_{ij,h}]^\alpha [\eta_{ij}]^\beta}{\sum_{l \in \mathcal{N}_i^k} [\tau_{il,h}]^\alpha [\eta_{il}]^\beta} = p_{ij,h}^k. \end{aligned}$$

□

**Theorem 1.** *Ant system is weakly-invariant.*<sup>1</sup>

<sup>1</sup>As already made clear, we refer here to the application of ant system to the traveling salesman problem and we consider the case in which the pheromone is initialized as prescribed by Definition 9.

*Proof.* Let us consider two generic instances  $I$  and  $\bar{I}$  such that

$$\bar{I} = fI, \text{ with } f > 0.$$

The theorem is proved by induction: We show that if at the generic iteration  $h$  some set of conditions  $\mathcal{C}$  holds, then the solutions generated for the two instances  $I$  and  $\bar{I}$  are the same and the set of conditions  $\mathcal{C}$  also holds for the following iteration  $h + 1$ . The proof is concluded by showing that  $\mathcal{C}$  holds for the very first iteration. With few minor modifications, this technique is adopted in the following for proving all theorems enunciated in the paper.

According to Definition 6, and taking into account Equation 1, it results:

$$\bar{\eta}_{ij} = \frac{1}{f}\eta_{ij}, \text{ for all } \langle i, j \rangle.$$

According to Lemma 1, if at the generic iteration  $h$ ,  $\bar{\tau}_{ij,h} = \frac{1}{f}\tau_{ij,h}$ , for all  $\langle i, j \rangle$ , then  $\bar{p}_{ij,h}^k = p_{ij,h}^k$ , for all  $\langle i, j \rangle$ . Under Hypothesis 1,

$$\bar{T}_h^k = T_h^k, \text{ for all } k = 1, \dots, m,$$

and therefore, according to Equation 2,

$$\bar{C}_h^k = fC_h^k, \text{ for all } k = 1, \dots, m.$$

According to Equation 3:

$$\begin{aligned} \bar{\Delta}_{ij,h}^k &= \begin{cases} 1/\bar{C}_h^k, & \text{if } \langle i, j \rangle \in \bar{T}_h^k; \\ 0, & \text{otherwise;} \end{cases} = \begin{cases} 1/fC_h^k, & \text{if } \langle i, j \rangle \in \bar{T}_h^k = T_h^k; \\ 0/f, & \text{otherwise;} \end{cases} \\ &= \frac{1}{f} \begin{cases} 1/C_h^k, & \text{if } \langle i, j \rangle \in T_h^k; \\ 0, & \text{otherwise;} \end{cases} = \frac{1}{f}\Delta_{ij,h}^k, \end{aligned}$$

and therefore, for any arc  $\langle i, j \rangle$  :

$$\begin{aligned} \bar{\tau}_{ij,h+1} &= (1 - \rho)\bar{\tau}_{ij,h} + \sum_{k=1}^m \bar{\Delta}_{ij,h}^k = (1 - \rho)\frac{1}{f}\tau_{ij,h} + \sum_{k=1}^m \frac{1}{f}\Delta_{ij,h}^k \\ &= (1 - \rho)\frac{1}{f}\tau_{ij,h} + \frac{1}{f} \sum_{k=1}^m \Delta_{ij,h}^k = \frac{1}{f} \left( (1 - \rho)\tau_{ij,h} + \sum_{k=1}^m \Delta_{ij,h}^k \right) = \frac{1}{f}\tau_{ij,h+1}. \end{aligned}$$

In order to provide a basis for the above defined induction and therefore to conclude the proof, it is sufficient to observe that at the first iteration  $h = 1$ , the pheromone is initialized as:

$$\bar{\tau}_{ij,1} = \frac{m}{\bar{C}_{nn}} = \frac{m}{fC_{nn}} = \frac{1}{f}\tau_{ij,1}, \text{ for all } \langle i, j \rangle.$$

□

*Remark 3.* Theorem 1 holds true for any way of initializing the pheromone, provided that for any two instances  $\bar{I}$  and  $I$  such that  $\bar{I} = fI$ ,  $\bar{\tau}_{ij,1} = \frac{1}{f}\tau_{ij,1}$ , for all  $\langle i, j \rangle$ .

*Remark 4.* Theorem 1 extends to the application of ant system to problems other than the traveling salesman problem, provided that the initialization of the pheromone is performed as prescribed in Remark 3 and for any two instances  $\bar{I}$  and  $I$  such that  $\bar{I} = fI$ , with  $f > 0$ , there exists a coefficient  $g > 0$  such that  $\bar{\eta}_{ij} = g\eta_{ij}$ , for all  $\langle i, j \rangle$ . In particular, it is worth pointing out here that one notable case in which this last condition is satisfied is when  $\bar{\eta}_{ij} = \eta_{ij} = 0$ , for all  $\langle i, j \rangle$ , that is, when no heuristic information is used.

## Strongly-invariant ant system

A strongly-invariant version of ant system (*siAS*) can be defined. For definiteness, we present here a version of *siAS* for the traveling salesman problem.

**Definition 10 (Strongly-invariant heuristic information).** When solving the traveling salesman problem, the heuristic information  $\eta_{ij}$  is

$$\eta_{ij} = \frac{C^{nn}}{nc_{ij}}, \text{ for all } \langle i, j \rangle. \quad (4)$$

where  $c_{ij}$  is the cost of traveling from city  $i$  to city  $j$ ,  $n$  is the number of cities, and  $C^{nn}$  is the cost of the solution  $T^{nn}$  obtained by the nearest-neighbor heuristic.

**Definition 11 (Strongly-invariant pheromone update rule).** The pheromone is updated using the same rule given in Definition 7, with the only difference that  $\Delta_{ij,h}^k$  is given by:

$$\Delta_{ij,h}^k = \begin{cases} C^{nn}/mC_h^k, & \text{if } \langle i, j \rangle \in T_h^k; \\ 0, & \text{otherwise;} \end{cases}$$

where  $C^{nn}$  is the cost of the solution  $T^{nn}$  obtained by the nearest-neighbor heuristic and  $m$  is the number of ants generated at each iteration.

**Definition 12 (Strongly-invariant pheromone initialization).** At the first iteration  $h = 1$ , the pheromone on all arcs is initialized to the value:  $\tau_{ij,1} = \tau_{init} = 1$ , for all  $\langle i, j \rangle$ .

**Definition 13 (Strongly-invariant ant system).** The strongly-invariant ant system (*siAS*) is a variation of ant system. It shares with ant system the random proportional rule for the construction of solutions, but in *siAS* the heuristic values are set as in Definition 10, the pheromone is initialized according to Definition 12, and the update is performed according to Definition 11.

**Theorem 2.** *Ant system and siAS are functionally equivalent.*

*Proof.* In this proof, a tilde placed above a symbol indicates that it refers to *siAS*: For example, if  $\tau_{ij,h}$  is the amount of pheromone on arc  $\langle i, j \rangle$  at iteration  $h$  in ant system, then  $\tilde{\tau}_{ij,h}$  is the amount of pheromone on arc  $\langle i, j \rangle$  at iteration  $h$  in *siAS*.

Let us consider a generic instance  $I$ . According to Definitions 6 and 10,  $\tilde{\eta}_{ij} = g_2\eta_{ij}$ , with  $g_2 = C^{nn}/n$ . According to Lemma 1, if at the generic iteration  $h$ ,  $\tilde{\tau}_{ij,h} = g_1\tau_{ij,h}$ , for all  $\langle i, j \rangle$ , then  $\tilde{p}_{ij,h}^k = p_{ij,h}^k$ , for all  $\langle i, j \rangle$ . Under Hypothesis 1,  $\tilde{T}_h^k = T_h^k$ , for all  $k = 1, \dots, m$  and therefore,  $\tilde{C}_h^k = C_h^k$ , for all  $k = 1, \dots, m$ .

Obviously, for what concerns the solution returned by the nearest-neighbor heuristic:  $\tilde{T}^{nn} = T^{nn}$  and  $\tilde{C}^{nn} = C^{nn}$ . Now, let  $g_1 = C^{nn}/m$ . According to Definitions 7 and 11,

$$\tilde{\Delta}_{ij,h}^k = \begin{cases} \tilde{C}^{nn}/m\tilde{C}_h^k, & \text{if } \langle i, j \rangle \in \tilde{T}_h^k; \\ 0, & \text{otherwise;} \end{cases} = \frac{C^{nn}}{m} \begin{cases} 1/C_h^k, & \text{if } \langle i, j \rangle \in T_h^k; \\ 0, & \text{otherwise;} \end{cases} = g_1\Delta_{ij,h}^k.$$

Therefore, for any arc  $\langle i, j \rangle$ :

$$\tilde{\tau}_{ij,h+1} = (1 - \rho)\tilde{\tau}_{ij,h} + \sum_{k=1}^m \tilde{\Delta}_{ij,h}^k = (1 - \rho)g_1\tau_{ij,h} + \sum_{k=1}^m g_1\Delta_{ij,h}^k = g_1\tau_{ij,h+1}.$$

The proof is completed by observing that, according to Definitions 9 and 12, at the first iteration  $h = 1$ :

$$\tilde{\tau}_{ij,1} = 1 = \frac{C^{nn}}{m} \frac{m}{C^{nn}} = g_1\tau_{ij,1}, \text{ for all } \langle i, j \rangle.$$

□



**Theorem 3.** *siAS is strongly-invariant.*

*Proof.* The weak invariance of *siAS* follows trivially from Theorems 1 and 2: Since *siAS* is functionally equivalent to ant system and the latter is weakly-invariant, also *siAS* is weakly-invariant. To prove strong invariance, we need to show that the heuristic information and the pheromone are invariant on all arcs.

Let us consider two instances  $I$  and  $\bar{I}$  such that  $\bar{I} = fI$ . According to Definition 10, and taking into account Equation 1, it results:  $\bar{\eta}_{ij} = \bar{C}^{nn}/n\bar{c}_{ij} = fC^{nn}/nfc_{ij} = C^{nn}/nc_{ij} = \eta_{ij}$ , for all  $\langle i, j \rangle$ . Under Hypothesis 1, if  $\bar{\tau}_{ij,h} = \tau_{ij,h}$ , for all  $\langle i, j \rangle$  at a generic iteration  $h$ , then  $\bar{\tau}_{ij,h+1} = \tau_{ij,h+1}$ , for all  $\langle i, j \rangle$ . The proof is completed by observing that at the first iteration  $h = 1$ ,  $\bar{\tau}_{ij,1} = \tau_{ij,1} = 1$ , for all  $\langle i, j \rangle$ .  $\square$

*Remark 5.* In the above definition of *siAS*, the nearest-neighbor heuristic has been adopted for generating a reference solution, the cost of which is then used for normalizing the cost of the solutions found by *siAS*. Any other algorithm could be used instead, provided that the solution it returns does not depend on the scale of the problem.

*Remark 6.* It is worth noting here that the presence of the term  $n$  in the denominator of the left hand side of Equation 4 is not needed for obtaining an invariant heuristic information. It has been included for achieving another property. Indeed,  $\eta_{ij}$  as defined in Equation 4 assumes values that do not depend on the size of the instance under analysis—that is, on the number  $n$  of cities. If this term were not present, since the numerator  $C^{nn}$  grows with  $n$ ,  $\eta_{ij}$  would have been relatively larger in large instances and smaller in small ones.

*Remark 7.* Similarly, it should be noticed that by initializing the pheromone to  $\tau_{ij,1} = \tau_{mit} = 1/m$ , for all  $\langle i, j \rangle$ , and by defining  $\Delta_{ij,h}^k$  as:

$$\Delta_{ij,h}^k = \begin{cases} C^{nn}/C_h^k, & \text{if } \langle i, j \rangle \in T_h^k; \\ 0, & \text{otherwise;} \end{cases}$$

one would have obtained nonetheless an invariant algorithm. The advantage of the formulation given in Definitions 11 and 12 is that the magnitude of the pheromone deposited on the arcs does not depend on the number  $m$  of ants considered.

## 4 $\mathcal{MAX-MIN}$ ant system

The results given for ant system can be extended to  $\mathcal{MAX-MIN}$  ant system [11, 12]. The characterizing element of  $\mathcal{MAX-MIN}$  ant system is the fact that the pheromone values are constrained between a minimum and a maximum, which possibly change iteration by iteration.

**Definition 14 (Pheromone trail limits).** At iteration  $h + 1$ , the pheromone value  $\tau_{ij,h+1}$  on a generic arc  $\langle i, j \rangle$  is constrained:

$$\tau_h^{min} \leq \tau_{ij,h+1} \leq \tau_h^{max},$$

with  $\tau_h^{max} = 1/\rho C_h^{bs}$  and  $\tau_h^{min} = a\tau_h^{max}$ , where  $C_h^{bs}$  is the best solution found up to and including iteration  $h$ , and  $a$  is a parameter.

*Remark 8.* The following notation will be adopted:

$$[x]_{min}^{max} = \begin{cases} min, & \text{if } x < min; \\ x, & \text{if } min \leq x \leq max; \\ max, & \text{if } x > max. \end{cases}$$

It can be easily shown that, if  $g > 0$ ,

$$[g \cdot x]_{g \cdot \min}^{g \cdot \max} = g[x]_{\min}^{\max}.$$

This property will be used in the following.

**Definition 15 (Pheromone update rule).** If  $\tau_{ij,h}$  is the value of the pheromone on arc  $\langle i, j \rangle$  at the current iteration  $h$ , the value of the pheromone at iteration  $h + 1$  is given by:

$$\tau_{ij,h+1} = \left[ (1 - \rho)\tau_{ij,h} + \Delta_{ij,h}^{\text{best}} \right]_{\tau_h^{\min}}^{\tau_h^{\max}}, \quad (5)$$

where  $\rho$  is the evaporation rate. The quantity  $\Delta_{ij,h}^{\text{best}}$  might be given by two different equations depending on which update mode is performed: either *best-so-far* update or *iteration-best* update. In the *best-so-far* update:

$$\Delta_{ij,h}^{\text{best}} = \begin{cases} 1/C_h^{\text{bs}}, & \text{if } \langle i, j \rangle \in T_h^{\text{bs}}; \\ 0, & \text{otherwise;} \end{cases} \quad (6)$$

where  $T_h^{\text{bs}}$  is the best solution found up to and including iteration  $h$ , and  $C_h^{\text{bs}}$  is its cost. In the *iteration-best* update:

$$\Delta_{ij,h}^{\text{best}} = \begin{cases} 1/C_h^{\text{ib}}, & \text{if } \langle i, j \rangle \in T_h^{\text{ib}}; \\ 0, & \text{otherwise;} \end{cases} \quad (7)$$

where  $T_h^{\text{ib}}$  is the best solution found in iteration  $h$ , and  $C_h^{\text{ib}}$  is its cost.

*Remark 9.* Whether a *best-so-far* or an *iteration-best* update is to be performed at a given iteration  $h$  is a design choice. In the typical implementation of  $\mathcal{MAX}\text{-MLN}$  ant system, mostly the *iteration-best* update is adopted in the initial iterations and the frequency with which the *best-so-far* update is employed increases iteration after iteration.

**Definition 16 ( $\mathcal{MAX}\text{-MLN}$  ant system).**  $\mathcal{MAX}\text{-MLN}$  ant system is an ACO algorithm in which solutions are constructed according to the random proportional rule given in Definition 5, and the pheromone is updated according to Definition 15. The evaporation rate  $\rho$ , the number of ants  $m$ , the exponents  $\alpha$  and  $\beta$ , and the factor  $a$  are parameters of the algorithm.

**Definition 17 (Nearest-neighbor pheromone initialization).** At the first iteration  $h = 1$ , the pheromone on all arcs is initialized to the value:

$$\tau_{ij,1} = \tau_{\text{mit}} = \frac{1}{\rho C^{\text{nn}}}, \text{ for all } \langle i, j \rangle,$$

where  $C^{\text{nn}}$  is the cost of the solution  $T^{\text{nn}}$  obtained by the nearest-neighbor heuristic.

**Theorem 4.**  $\mathcal{MAX}\text{-MLN}$  ant system is weakly-invariant.<sup>2</sup>

*Proof.* The proof follows the one given for Theorem 1. Let us consider two generic instances  $I$  and  $\bar{I}$  such that  $\bar{I} = fI$ , with  $f > 0$ . It results that  $\bar{\eta}_{ij} = \frac{1}{f}\eta_{ij}$ , for all  $\langle i, j \rangle$ . Let us assume that, at the beginning of the generic iteration  $h$ ,  $\bar{T}_{h-1}^{\text{bs}} = T_{h-1}^{\text{bs}}$  and  $\bar{\tau}_{ij,h} = \frac{1}{f}\tau_{ij,h}$ , for all  $\langle i, j \rangle$ . According to Lemma 1,  $\bar{p}_{ij,h}^k = p_{ij,h}^k$ , for all  $\langle i, j \rangle$ . Under Hypothesis 1,  $\bar{T}_h^k = T_h^k$ , for all  $k = 1, \dots, m$ , and therefore,  $\bar{C}_h^k = fC_h^k$ , for all  $k = 1, \dots, m$ . In particular,  $\bar{T}_h^{\text{ib}} = T_h^{\text{ib}}$ , and  $\bar{C}_h^{\text{ib}} = fC_h^{\text{ib}}$ . Moreover, whether or not an improvement is made on the *best-so-far* solution,  $\bar{T}_h^{\text{bs}} = T_h^{\text{bs}}$ . Indeed, since  $\bar{T}_{h-1}^{\text{bs}} = T_{h-1}^{\text{bs}}$ , then  $\bar{C}_{h-1}^{\text{bs}} = fC_{h-1}^{\text{bs}}$ . If  $C_h^{\text{ib}} < C_{h-1}^{\text{bs}}$ , then also  $\bar{C}_h^{\text{ib}} < \bar{C}_{h-1}^{\text{bs}}$ , and  $\bar{T}_h^{\text{bs}} = \bar{T}_h^{\text{ib}} = T_h^{\text{ib}} = T_h^{\text{bs}}$ . On the other hand, if  $C_h^{\text{ib}} \geq C_{h-1}^{\text{bs}}$ , then also  $\bar{C}_h^{\text{ib}} \geq \bar{C}_{h-1}^{\text{bs}}$ , and  $\bar{T}_h^{\text{bs}} = \bar{T}_{h-1}^{\text{bs}} = T_{h-1}^{\text{bs}} = T_h^{\text{bs}}$ .

<sup>2</sup>Also in this case, for definiteness, we refer to the application of  $\mathcal{MAX}\text{-MLN}$  ant system to the traveling salesman problem and we consider the case in which the pheromone is initialized as prescribed by Definition 17.

If at step  $h$  a *best-so-far* update is applied, according to Equation 6:

$$\begin{aligned}\bar{\Delta}_{ij,h}^{best} &= \begin{cases} 1/\bar{C}_h^{bs}, & \text{if } \langle i, j \rangle \in \bar{T}_h^{bs}; \\ 0, & \text{otherwise;} \end{cases} = \begin{cases} 1/fC_h^{bs}, & \text{if } \langle i, j \rangle \in \bar{T}_h^{bs} = T_h^{bs}; \\ 0/f, & \text{otherwise;} \end{cases} \\ &= \frac{1}{f} \begin{cases} 1/C_h^{bs}, & \text{if } \langle i, j \rangle \in T_h^{bs}; \\ 0, & \text{otherwise;} \end{cases} = \frac{1}{f} \Delta_{ij,h}^{best}.\end{aligned}$$

Similarly, if an *iteration-best* update is applied, according to Equation 7:

$$\begin{aligned}\bar{\Delta}_{ij,h}^{best} &= \begin{cases} 1/\bar{C}_h^{ib}, & \text{if } \langle i, j \rangle \in \bar{T}_h^{ib}; \\ 0, & \text{otherwise;} \end{cases} = \begin{cases} 1/fC_h^{ib}, & \text{if } \langle i, j \rangle \in \bar{T}_h^{ib} = T_h^{ib}; \\ 0/f, & \text{otherwise;} \end{cases} \\ &= \frac{1}{f} \begin{cases} 1/C_h^{ib}, & \text{if } \langle i, j \rangle \in T_h^{ib}; \\ 0, & \text{otherwise;} \end{cases} = \frac{1}{f} \Delta_{ij,h}^{best}.\end{aligned}$$

In both cases,

$$\bar{\tau}_h^{max} = \frac{1}{\rho \bar{C}_h^{bs}} = \frac{1}{f \rho C_h^{bs}} = \frac{1}{f} \tau_h^{max},$$

and therefore,

$$\bar{\tau}_h^{min} = a \bar{\tau}_h^{max} = \frac{a}{f} \tau_h^{max} = \frac{1}{f} \tau_h^{min}.$$

It follows that,

$$\begin{aligned}\bar{\tau}_{ij,h+1} &= \left[ (1 - \rho) \bar{\tau}_{ij,h} + \bar{\Delta}_{ij,h}^{best} \right]_{\bar{\tau}_h^{min}}^{\bar{\tau}_h^{max}} = \left[ (1 - \rho) \frac{1}{f} \tau_{ij,h} + \frac{1}{f} \Delta_{ij,h}^{best} \right]_{\frac{1}{f} \tau_h^{min}}^{\frac{1}{f} \tau_h^{max}} \\ &= \left[ \frac{1}{f} \left( (1 - \rho) \tau_{ij,h} + \Delta_{ij,h}^{best} \right) \right]_{\frac{1}{f} \tau_h^{min}}^{\frac{1}{f} \tau_h^{max}} = \frac{1}{f} \left[ (1 - \rho) \tau_{ij,h} + \Delta_{ij,h}^{best} \right]_{\tau_h^{min}}^{\tau_h^{max}} = \frac{1}{f} \tau_{ij,h+1}.\end{aligned}$$

The proof is completed by observing that at the first iteration  $h = 1$ , the pheromone is initialized as:

$$\bar{\tau}_{ij,1} = \frac{1}{\rho \bar{C}^{nn}} = \frac{1}{f \rho C^{nn}} = \frac{1}{f} \tau_{ij,1},$$

and the initial *best-so-far* solutions are  $\bar{T}_0^{bs} = \bar{T}^{nn} = T^{nn} = T_0^{bs}$ .  $\square$

## Strongly-invariant $\mathcal{MAX}$ - $\mathcal{MIN}$ ant system

A strongly-invariant version of  $\mathcal{MAX}$ - $\mathcal{MIN}$  ant system (*siMMAS*) can be defined. For definiteness, we present here a version of *siMMAS* for the traveling salesman problem.

**Definition 18 (Strongly-invariant pheromone update rule).** The pheromone is updated as in Definition 15, with the difference that in the *best-so-far* update:

$$\Delta_{ij,h}^{best} = \begin{cases} \rho C^{nn} / C_h^{bs}, & \text{if } \langle i, j \rangle \in T_h^{bs}; \\ 0, & \text{otherwise;} \end{cases} \quad (8)$$

and in the *iteration-best* update:

$$\Delta_{ij,h}^{best} = \begin{cases} \rho C^{nn} / C_h^{ib}, & \text{if } \langle i, j \rangle \in T_h^{ib}; \\ 0, & \text{otherwise;} \end{cases} \quad (9)$$

where  $C^{nn}$  is the cost of the solution  $T^{nn}$  obtained by the nearest-neighbor heuristic and  $\rho$  is the evaporation rate.

**Definition 19 (Strongly-invariant pheromone initialization).** At the first iteration  $h = 1$ , the pheromone on all arcs is initialized to the value:  $\tau_{ij,1} = \tau_{init} = 1$ , for all  $\langle i, j \rangle$ .

**Definition 20 (Strongly-invariant pheromone trail limits).** At iteration  $h + 1$ , the value  $\tau_{ij,h+1}$  of the pheromone on a generic arc  $\langle i, j \rangle$  is constrained:  $\tau_h^{min} \leq \tau_{ij,h+1} \leq \tau_h^{max}$ , with  $\tau_h^{max} = C^{nn}/C_h^{bs}$  and  $\tau_h^{min} = a\tau_h^{max}$ , where  $C^{nn}$  is the cost of the solution  $T^{nn}$  found by the nearest-neighbor algorithm,  $C_h^{bs}$  is the best solution found up to and including iteration  $h$ , and  $a$  is a parameter.

**Definition 21 (Strongly-invariant  $\mathcal{MAX}\text{-}\mathcal{MLN}$  ant system).** The strongly-invariant  $\mathcal{MAX}\text{-}\mathcal{MLN}$  ant system (*siMMAS*) is a minor variation of  $\mathcal{MAX}\text{-}\mathcal{MLN}$  ant system. It shares with  $\mathcal{MAX}\text{-}\mathcal{MLN}$  ant system the random proportional rule given in Definition 5, the heuristic values are set as in Definition 10, the pheromone is initialized according to Definition 19 and limited according to Definition 20, and the update is performed according to Definition 18.

**Theorem 5.**  *$\mathcal{MAX}\text{-}\mathcal{MLN}$  ant system and siMMAS are functionally equivalent.*

*Proof.* As in the proof of Theorem 2, a tilde placed above a symbol indicates that the latter refers to *siMMAS*. Let us consider a generic instance  $I$ . According to Definitions 6 and 10,  $\tilde{\eta}_{ij} = g_2\eta_{ij}$ , with  $g_2 = C^{nn}/n$ . According to Lemma 1, if at the generic iteration  $h$ ,  $\tilde{\tau}_{ij,h} = g_1\tau_{ij,h}$ , for all  $\langle i, j \rangle$ , then  $\tilde{p}_{ij,h}^k = p_{ij,h}^k$ , for all  $\langle i, j \rangle$ . Under Hypothesis 1,  $\tilde{T}_h^k = T_h^k$ , for all  $k = 1, \dots, m$  and therefore,  $\tilde{C}_h^k = C_h^k$ , for all  $k = 1, \dots, m$ . In particular,  $\tilde{T}_h^{ib} = T_h^{ib}$ , and  $\tilde{C}_h^{ib} = C_h^{ib}$ . Moreover, whether or not an improvement is made on the *best-so-far* solution,  $\tilde{T}_h^{bs} = T_h^{bs}$ —see the proof of Theorem 4.

For what concerns the solution returned by the nearest-neighbor heuristic:  $\tilde{T}^{nn} = T^{nn}$  and  $\tilde{C}^{nn} = C^{nn}$ . Now, let  $g_1 = \rho C^{nn}$ . If at step  $h$  a *best-so-far* update is applied, according to Equation 8:

$$\tilde{\Delta}_{ij,h}^{best} = \begin{cases} \rho\tilde{C}^{nn}/\tilde{C}_h^{bs}, & \text{if } \langle i, j \rangle \in \tilde{T}_h^{bs}; \\ 0, & \text{otherwise;} \end{cases} = \rho C^{nn} \begin{cases} 1/C_h^{bs}, & \text{if } \langle i, j \rangle \in T_h^{bs}; \\ 0, & \text{otherwise;} \end{cases} = g_1\Delta_{ij,h}^{best}.$$

Similarly, if an *iteration-best* update is applied, according to Equation 9:

$$\tilde{\Delta}_{ij,h}^{best} = \begin{cases} \rho\tilde{C}^{nn}/\tilde{C}_h^{ib}, & \text{if } \langle i, j \rangle \in \tilde{T}_h^{ib}; \\ 0, & \text{otherwise;} \end{cases} = \rho C^{nn} \begin{cases} 1/C_h^{ib}, & \text{if } \langle i, j \rangle \in T_h^{ib}; \\ 0, & \text{otherwise;} \end{cases} = g_1\Delta_{ij,h}^{best}.$$

In both cases, according to Definitions 14 and 20,  $\bar{\tau}_h^{max} = C^{nn}/C_h^{bs} = \rho C^{nn}/\rho C_h^{bs} = g_1\tau_h^{max}$  and  $\bar{\tau}_h^{min} = a\bar{\tau}_h^{max} = ag_1\tau_h^{max} = g_1\tau_h^{min}$ . It follows that, for all  $\langle i, j \rangle$ :

$$\tilde{\tau}_{ij,h+1} = \left[ (1 - \rho)\tilde{\tau}_{ij,h} + \tilde{\Delta}_{ij,h}^{best} \right]_{\bar{\tau}_h^{min}}^{\bar{\tau}_h^{max}} = \left[ (1 - \rho)g_1\tau_{ij,h} + g_1\Delta_{ij,h}^{best} \right]_{g_1\tau_h^{min}}^{g_1\tau_h^{max}} = g_1\tau_{ij,h+1}.$$

The proof is completed by observing that at the first iteration  $h = 1$ , the pheromone is initialized as  $\tilde{\tau}_{ij,1} = 1 = \rho C^{nn}/\rho C^{nn} = g_1\tau_{ij,1}$ , and the initial *best-so-far* solutions are  $\tilde{T}_0^{bs} = \tilde{T}^{nn} = T^{nn} = T_0^{bs}$ .  $\square$

**Theorem 6.** *siMMAS is strongly-invariant.*

*Proof.* The weak invariance of *siMMAS* follows trivially from Theorems 4 and 5. In order to show that the heuristic information and the pheromone are invariant on all arcs, let us consider two instances  $I$  and  $\bar{I}$  such that  $\bar{I} = fI$ . According to Definition 10, and taking into account Equation 1, it results:  $\bar{\eta}_{ij} = \bar{C}^{nn}/n\bar{c}_{ij} = fC^{nn}/nf c_{ij} = C^{nn}/nc_{ij} = \eta_{ij}$ . Under Hypothesis 1, if  $\bar{\tau}_{ij,h} = \tau_{ij,h}$ , for all  $\langle i, j \rangle$  at a generic iteration  $h$ , and if  $\bar{T}_{h-1}^{bs} = T_{h-1}^{bs}$ , then  $\bar{\tau}_{ij,h+1} = \tau_{ij,h+1}$ , for all  $\langle i, j \rangle$ . The proof is completed by observing that at the first iteration  $h = 1$ ,  $\bar{\tau}_{ij,1} = \tau_{ij,1} = 1$ , for all  $\langle i, j \rangle$ , and the initial *best-so-far* solutions are  $\bar{T}_0^{bs} = \bar{T}^{nn} = T^{nn} = T_0^{bs}$ .  $\square$

*Remark 10.* Also in the case of *siMMAS*, the nearest-neighbor heuristic has been adopted for generating a reference solution. Any other algorithm could be used instead, provided that the solution it returns does not depend on the scale of the problem.

## 5 Ant colony system

The weak invariance property holds also for ant colony system [10]. In ant colony system, the concept of *local* pheromone update is introduced: When an ant traverses an arc while constructing a solution, the pheromone on that arc is reduced. In order to describe this feature, a slightly modified notation is needed.

**Definition 22 (Local pheromone update rule).** At the generic iteration  $h$ , after ant  $k$  has built the solution  $T_h^k$ , the pheromone is modified according to:

$$\tau_{ij,h}^{(k)} = \begin{cases} (1 - \xi)\tau_{ij,h}^{(k-1)} + \xi\tau_{init}, & \text{if } \langle i, j \rangle \in T_h^k; \\ \tau_{ij,h}^{(k-1)}, & \text{otherwise;} \end{cases}$$

where  $\xi$  is a parameter and  $\tau_{init}$  is the initial value of the pheromone—see Definition 26. Moreover,  $\tau_{ij,h}^{(k-1)}$  is the value of the pheromone on arc  $\langle i, j \rangle$  at iteration  $h$  after the first  $k - 1$  ants have constructed their respective solution and before the transit of ant  $k$ . On the other hand,  $\tau_{ij,h}^{(k)}$  is the value after ant  $k$  has transited.

**Definition 23 (Global pheromone update rule).** At each iteration  $h$ , after all  $m$  ants have built their solution and performed the local pheromone update, the pheromone on the arcs belonging to the best solution  $T_h^{bs}$  found up to and including iteration  $h$ , are reinforced:

$$\tau_{ij,h+1}^{(0)} = \begin{cases} (1 - \rho)\tau_{ij,h}^{(m)} + \rho\Delta_{ij,h}^{best}, & \text{if } \langle i, j \rangle \in T_h^{bs}; \\ \tau_{ij,h}^{(m)}, & \text{otherwise;} \end{cases}$$

where  $\Delta_{ij,h}^{best} = 1/C_h^{bs}$ , being  $C_h^{bs}$  the cost of  $T_h^{bs}$ . The quantity  $\tau_{ij,h}^{(m)}$  is the value of the pheromone on arc  $\langle i, j \rangle$  after all  $m$  ants have constructed their respective solution at iteration  $h$ , while  $\tau_{ij,h+1}^{(0)}$  is the quantity of pheromone on arc  $\langle i, j \rangle$  right before any ant starts building its solution at iteration  $h + 1$ .

**Definition 24 (Pseudorandom proportional rule).** At the generic iteration  $h$ , ant  $k$  which is in node  $i$  and has  $\mathcal{N}_i^k$  as set of feasible nodes, selects the node to be visited next according to the following rule: With a probability given by the parameter  $q_0$ , the ant moves to the feasible node that maximizes  $\tau_{il,h}^{(k-1)}[\eta_{il}]^\beta$ , where  $l \in \mathcal{N}_i^k$ ; with probability  $1 - q_0$  a node is selected according to the random proportional rule given in Definition 5 (with  $\alpha = 1$ ). In the framework of this paper, the following equivalent statement is more convenient:

$$p_{ij,h}^k = \begin{cases} q_0 + (1 - q_0) \frac{\tau_{ij,h}^{(k-1)}[\eta_{ij}]^\beta}{\sum_{l \in \mathcal{N}_i^k} \tau_{il,h}^{(k-1)}[\eta_{il}]^\beta}, & \text{if } j = \arg \max_{l \in \mathcal{N}_i^k} \tau_{il,h}^{(k-1)}[\eta_{il}]^\beta; \\ (1 - q_0) \frac{\tau_{ij,h}^{(k-1)}[\eta_{ij}]^\beta}{\sum_{l \in \mathcal{N}_i^k} \tau_{il,h}^{(k-1)}[\eta_{il}]^\beta}, & \text{otherwise;} \end{cases}$$

where  $\beta$  and  $q_0$  are parameters, with  $0 \leq q_0 \leq 1$ .

**Lemma 2.** *The pseudorandom proportional rule is invariant to concurrent linear transformations of the pheromone and of the heuristic information. Formally, for any two positive constants  $g_1$  and  $g_2$ ,*

$$\bar{\tau}_{ij,h}^{(k-1)} = g_1 \tau_{ij,h}^{(k-1)} \wedge \bar{\eta}_{ij} = g_2 \eta_{ij}, \text{ for all } \langle i, j \rangle \implies \bar{p}_{ij,h}^k = p_{ij,h}^k, \text{ for all } \langle i, j \rangle.$$

where  $\bar{p}_{ij,h}^k$  is obtained on the basis of  $\bar{\tau}_{ij,h}^{(k-1)}$  and  $\bar{\eta}_{ij}$ , according to Definition 24.

*Proof.* Indeed:

$$\begin{aligned} \bar{p}_{ij,h}^k &= \begin{cases} q_0 + (1 - q_0) \frac{\bar{\tau}_{ij,h}^{(k-1)} [\bar{\eta}_{ij}]^\beta}{\sum_{l \in \mathcal{N}_i^k} \bar{\tau}_{il,h}^{(k-1)} [\bar{\eta}_{il}]^\beta}, & \text{if } j = \arg \max_{l \in \mathcal{N}_i^k} \bar{\tau}_{il,h} [\bar{\eta}_{il}]^\beta; \\ (1 - q_0) \frac{\bar{\tau}_{ij,h}^{(k-1)} [\bar{\eta}_{ij}]^\beta}{\sum_{l \in \mathcal{N}_i^k} \bar{\tau}_{il,h}^{(k-1)} [\bar{\eta}_{il}]^\beta}, & \text{otherwise;} \end{cases} \\ &= \begin{cases} q_0 + (1 - q_0) \frac{g_1 \tau_{ij,h}^{(k-1)} [g_2 \eta_{ij}]^\beta}{\sum_{l \in \mathcal{N}_i^k} g_1 \tau_{il,h}^{(k-1)} [g_2 \eta_{il}]^\beta}, & \text{if } j = \arg \max_{l \in \mathcal{N}_i^k} g_1 \tau_{il,h} [g_2 \eta_{il}]^\beta; \\ (1 - q_0) \frac{g_1 \tau_{ij,h}^{(k-1)} [g_2 \eta_{ij}]^\beta}{\sum_{l \in \mathcal{N}_i^k} g_1 \tau_{il,h}^{(k-1)} [g_2 \eta_{il}]^\beta}, & \text{otherwise;} \end{cases} \\ &= \begin{cases} q_0 + (1 - q_0) \frac{g_1 [g_2]^\beta \tau_{ij,h}^{(k-1)} [\eta_{ij}]^\beta}{g_1 [g_2]^\beta \sum_{l \in \mathcal{N}_i^k} \tau_{il,h}^{(k-1)} [\eta_{il}]^\beta}, & \text{if } j = \arg \max_{l \in \mathcal{N}_i^k} g_1 [g_2]^\beta \tau_{il,h} [\eta_{il}]^\beta; \\ (1 - q_0) \frac{g_1 [g_2]^\beta \tau_{ij,h}^{(k-1)} [\eta_{ij}]^\beta}{g_1 [g_2]^\beta \sum_{l \in \mathcal{N}_i^k} \tau_{il,h}^{(k-1)} [\eta_{il}]^\beta}, & \text{otherwise;} \end{cases} \\ &= \begin{cases} q_0 + (1 - q_0) \frac{\tau_{ij,h}^{(k-1)} [\eta_{ij}]^\beta}{\sum_{l \in \mathcal{N}_i^k} \tau_{il,h}^{(k-1)} [\eta_{il}]^\beta}, & \text{if } j = \arg \max_{l \in \mathcal{N}_i^k} \tau_{il,h} [\eta_{il}]^\beta; \\ (1 - q_0) \frac{\tau_{ij,h}^{(k-1)} [\eta_{ij}]^\beta}{\sum_{l \in \mathcal{N}_i^k} \tau_{il,h}^{(k-1)} [\eta_{il}]^\beta}, & \text{otherwise;} \end{cases} \\ &= p_{ij,h}^k. \end{aligned}$$

□

**Definition 25 (Ant colony system).** Ant colony system is an ACO algorithm in which solutions are constructed according to the pseudorandom proportional rule given in Definition 24, and the pheromone is updated according to Definitions 22 and 23. The local and global evaporation rates  $\xi$  and  $\rho$ , the number of ants  $m$ , the exponent  $\beta$ , and the probability  $q_0$  are parameters of the algorithm.

**Definition 26 (Nearest-neighbor pheromone initialization).** At the first iteration  $h = 1$ , the pheromone on all  $\langle i, j \rangle$  is initialized to the value:

$$\tau_{ij,1}^{(0)} = \tau_{init} = \frac{1}{nC^{nn}},$$

where  $n$  is the number of nodes in the traveling salesman instance and  $C^{nn}$  is the cost of the solution  $T^{nn}$  obtained by the nearest-neighbor heuristic.

**Theorem 7.** *Ant colony system is weakly-invariant.*<sup>3</sup>

<sup>3</sup>Also in this case, for definiteness, we refer to the application of ant colony system to the traveling salesman problem and we consider the case in which the pheromone is initialized as prescribed by Definition 26.

*Proof.* The proof follows those given for Theorems 1 and 4. Let us consider two generic instances  $I$  and  $\bar{I}$  such that  $\bar{I} = fI$ , with  $f > 0$ . It results that  $\bar{\eta}_{ij} = \frac{1}{f}\eta_{ij}$ , for all  $\langle i, j \rangle$ . Moreover, from Definition 26:

$$\bar{\tau}_{init} = \frac{1}{n\bar{C}^{nn}} = \frac{1}{nfC^{nn}} = \frac{1}{f}\tau_{init}.$$

Let us assume that, at the beginning of the generic iteration  $h$ ,  $\bar{T}_{h-1}^{bs} = T_{h-1}^{bs}$  and  $\bar{\tau}_{ij,h}^{(0)} = \frac{1}{f}\tau_{ij,h}^{(0)}$ , for all  $\langle i, j \rangle$ . Let us consider the first ant that builds a solution at iteration  $h$ . According to Lemma 2,  $\bar{p}_{ij,h}^1 = p_{ij,h}^1$ , for all  $\langle i, j \rangle$ . Under Hypothesis 1,  $\bar{T}_h^1 = T_h^1$ . On the basis of Definition 22:

$$\begin{aligned} \bar{\tau}_{ij,h}^{(1)} &= \begin{cases} (1 - \xi)\bar{\tau}_{ij,h}^{(0)} + \xi\bar{\tau}_{init}, & \text{if } \langle i, j \rangle \in \bar{T}_h^1; \\ \bar{\tau}_{ij,h}^{(0)}, & \text{otherwise;} \end{cases} \\ &= \begin{cases} (1 - \xi)\frac{1}{f}\tau_{ij,h}^{(0)} + \xi\frac{1}{f}\tau_{init}, & \text{if } \langle i, j \rangle \in T_h^1; \\ \frac{1}{f}\tau_{ij,h}^{(0)}, & \text{otherwise;} \end{cases} \\ &= \frac{1}{f} \begin{cases} (1 - \xi)\tau_{ij,h}^{(0)} + \xi\tau_{init}, & \text{if } \langle i, j \rangle \in T_h^1; \\ \tau_{ij,h}^{(0)}, & \text{otherwise;} \end{cases} \\ &= \frac{1}{f}\tau_{ij,h}^{(1)} \end{aligned}$$

Under this condition, Lemma 2 applies also to the second ant of iteration  $h$ : It results  $\bar{p}_{ij,h}^2 = p_{ij,h}^2$ , for all  $\langle i, j \rangle$ . Therefore,  $\bar{T}_h^2 = T_h^2$  and finally  $\bar{\tau}_{ij,h}^{(2)} = \frac{1}{f}\tau_{ij,h}^{(2)}$ , for all  $\langle i, j \rangle$ . This procedure can be repeated for all  $m$  ants acting at generation  $h$  with the net result that  $\bar{\tau}_{ij,h}^{(m)} = \frac{1}{f}\tau_{ij,h}^{(m)}$  for all  $\langle i, j \rangle$ ,  $\bar{T}_h^k = T_h^k$ , for all  $k = 1, \dots, m$ , and therefore,  $\bar{C}_h^k = fC_h^k$ , for all  $k = 1, \dots, m$ . In particular,  $\bar{C}_h^{ib} = fC_h^{ib}$ . Moreover, whether or not an improvement is made on the *best-so-far* solution,  $\bar{T}_h^{bs} = T_h^{bs}$  and therefore  $\bar{C}_h^{bs} = fC_h^{bs}$ —see the proof of Theorem 4.

The global pheromone update takes place on the basis of:

$$\bar{\Delta}_{ij,h}^{best} = \frac{1}{\bar{C}_h^{bs}} = \frac{1}{fC_h^{bs}} = \frac{1}{f}\Delta_{ij,h}^{best}, \text{ if } \langle i, j \rangle \in T_h^{bs}.$$

It results:

$$\begin{aligned} \bar{\tau}_{ij,h+1}^{(0)} &= \begin{cases} (1 - \rho)\bar{\tau}_{ij,h}^{(m)} + \rho\bar{\Delta}_{ij,h}^{best}, & \text{if } \langle i, j \rangle \in \bar{T}_h^{bs}; \\ \bar{\tau}_{ij,h}^{(m)}, & \text{otherwise;} \end{cases} \\ &= \begin{cases} (1 - \rho)\frac{1}{f}\tau_{ij,h}^{(m)} + \rho\frac{1}{f}\Delta_{ij,h}^{best}, & \text{if } \langle i, j \rangle \in T_h^{bs}; \\ \frac{1}{f}\tau_{ij,h}^{(m)}, & \text{otherwise;} \end{cases} \\ &= \frac{1}{f} \begin{cases} (1 - \rho)\tau_{ij,h}^{(m)} + \rho\Delta_{ij,h}^{best}, & \text{if } \langle i, j \rangle \in T_h^{bs}; \\ \tau_{ij,h}^{(m)}, & \text{otherwise;} \end{cases} \\ &= \frac{1}{f}\tau_{ij,h+1}^{(0)}. \end{aligned}$$

The proof is completed by the fact that at the first iteration  $h = 1$ , the pheromone is initialized as:

$$\bar{\tau}_{ij,1}^{(0)} = \frac{1}{n\bar{C}^{nn}} = \frac{1}{nfC^{nn}} = \frac{1}{f}\tau_{ij,1}^{(0)}, \text{ for all } \langle i, j \rangle.$$

and the initial *best-so-far* solutions are  $\bar{T}_0^{bs} = \bar{T}^{nn} = T^{nn} = T_0^{bs}$ .  $\square$

## Strongly-invariant ant colony system

A strongly-invariant version of ant colony system (*siACS*) can be defined. For definiteness, we present here a version of *siACS* for the traveling salesman problem.

**Definition 27 (Strongly-invariant global pheromone update rule).** The global pheromone update is performed as in Definition 23, with the difference that

$$\Delta_{ij,h}^{best} = n \frac{C^{nn}}{C_h^{bs}},$$

where  $C^{nn}$  and  $C_h^{bs}$  are the costs of the nearest-neighbor solution  $T^{nn}$  and of the *best-so-far* solution  $T_h^{bs}$ , respectively, and  $n$  is the number of cities.

**Definition 28 (Strongly-invariant pheromone initialization).** At the first iteration  $h = 1$ , the pheromone on all arcs is initialized to the value  $\tau_{ij,1} = \tau_{init} = 1$ , for all  $\langle i, j \rangle$ .

**Definition 29 (Strongly-invariant ant colony system).** The strongly-invariant ant colony system (*siACS*) is a minor variation of ant colony system. It shares with ant colony system the pseudorandom proportional rule for the construction of solutions, the heuristic values are set as in Definition 10, the pheromone is initialized according to Definition 28 and the local and global pheromone update are performed according to Definitions 22 and 27, respectively.

**Theorem 8.** *Ant colony system and siACS are functionally equivalent.*

*Proof.* As in the proof of Theorems 2 and 5, a tilde placed above a symbol indicates that the latter refers to *siACS*. Let us consider a generic instance  $I$ . According to Definitions 6 and 10,  $\tilde{\eta}_{ij} = g_2 \eta_{ij}$ , with  $g_2 = C^{nn}/n$ . According to Lemma 2 and under Hypothesis 1, if at the beginning of the generic iteration  $h$ ,  $\tilde{\tau}_{ij,h}^{(0)} = g_1 \tau_{ij,h}^{(0)}$ , for all  $\langle i, j \rangle$ , then,  $\tilde{p}_{ij,h}^k = p_{ij,h}^k$ , for all  $\langle i, j \rangle$  and for all  $k = 1, \dots, m$ . Further, for all  $k = 1, \dots, m$ ,  $\tilde{\tau}_{ij,h}^{(k)} = g_1 \tau_{ij,h}^{(k)}$ ,  $\tilde{T}_h^k = T_h^k$ , and therefore  $\tilde{C}_h^k = C_h^k$ —see also the proof of Theorem 7. In particular,  $\tilde{\tau}_{ij,h}^{(m)} = g_1 \tau_{ij,h}^{(m)}$ . Moreover,  $\tilde{T}_h^{ib} = T_h^{ib}$  and  $\tilde{C}_h^{ib} = C_h^{ib}$ . Finally, whether or not an improvement is made on the *best-so-far* solution,  $\tilde{T}_h^{bs} = T_h^{bs}$ —see the proof of Theorem 4.

For what concerns the solution returned by the nearest-neighbor heuristic:  $\tilde{T}^{nn} = T^{nn}$  and  $\tilde{C}^{nn} = C^{nn}$ . Now, let  $g_1 = nC^{nn}$ . The global pheromone update takes place on the basis of the quantities:

$$\tilde{\Delta}_{ij,h}^{best} = \frac{n\tilde{C}^{nn}}{\tilde{C}_h^{bs}} = nC^{nn} \frac{1}{C_h^{bs}} = g_1 \tilde{\Delta}_{ij,h}^{best}, \text{ if } \langle i, j \rangle \in T_h^{bs}.$$

It results:

$$\begin{aligned} \tilde{\tau}_{ij,h+1}^{(0)} &= \begin{cases} (1 - \rho)\tilde{\tau}_{ij,h}^{(m)} + \rho\tilde{\Delta}_{ij,h}^{best}, & \text{if } \langle i, j \rangle \in \tilde{T}_h^{bs}; \\ \tilde{\tau}_{ij,h}^{(m)}, & \text{otherwise;} \end{cases} = \begin{cases} (1 - \rho)g_1\tau_{ij,h}^{(m)} + \rho g_1\Delta_{ij,h}^{best}, & \text{if } \langle i, j \rangle \in T_h^{bs}; \\ g_1\tau_{ij,h}^{(m)}, & \text{otherwise;} \end{cases} \\ &= g_1 \begin{cases} (1 - \rho)\tau_{ij,h}^{(m)} + \rho\Delta_{ij,h}^{best}, & \text{if } \langle i, j \rangle \in T_h^{bs}; \\ \tau_{ij,h}^{(m)}, & \text{otherwise;} \end{cases} = g_1\tau_{ij,h+1}^{(0)}. \end{aligned}$$

The proof is completed by the fact that at the first iteration  $h = 1$ , the pheromone is initialized as:

$$\tilde{\tau}_{ij,1}^{(0)} = 1 = \frac{nC^{nn}}{nC^{nn}} = g_1\tau_{ij,1}^{(0)}, \text{ for all } \langle i, j \rangle.$$

and the initial *best-so-far* solutions are  $\tilde{T}_0^{bs} = \tilde{T}^{nn} = T^{nn} = T_0^{bs}$ .  $\square$

**Theorem 9.** *siACS is strongly-invariant.*



*Proof.* The weak invariance of *siACS* follows trivially from Theorems 7 and 8. In order to show that the heuristic information and the pheromone are invariant on all arcs, let us consider two instances  $I$  and  $\bar{I}$  such that  $\bar{I} = fI$ . According to Definition 10, and taking into account Equation 1, it results:  $\bar{\eta}_{ij} = \bar{C}^{nn}/n\bar{c}_{ij} = fC^{nn}/nfc_{ij} = C^{nn}/nc_{ij} = \eta_{ij}$ . Under Hypothesis 1, if  $\bar{\tau}_{ij,h}^{(0)} = \tau_{ij,h}^{(0)}$ , for all  $\langle i, j \rangle$  at a generic iteration  $h$ , and if  $\bar{T}_{h-1}^{bs} = T_{h-1}^{bs}$ , then  $\bar{\tau}_{ij,h+1}^{(0)} = \tau_{ij,h+1}^{(0)}$ , for all  $\langle i, j \rangle$ . The proof is completed by observing that at the first iteration  $h = 1$ ,  $\bar{\tau}_{ij,1}^{(0)} = \tau_{ij,1}^{(0)} = 1$ , for all  $\langle i, j \rangle$ , and the initial *best-so-far* solutions are  $\bar{T}_0^{bs} = \bar{T}^{nn} = T^{nn} = T_0^{bs}$ .  $\square$

*Remark 11.* Also in the case of *siACS*, the nearest-neighbor heuristic has been adopted for generating a reference solution. Any other algorithm could be used instead, provided that the solution this algorithm returns does not depend on the scale of the problem.

## 6 Empirical analysis

An analysis of the performance of the three strongly-invariant algorithms proposed in the paper—namely *siAS*, *siMMAS*, and *siACS*—would be neither particularly novel nor interesting. Indeed, we have formally proved in Sections 3, 4, and 5 that *siAS*, *siMMAS*, and *siACS* are *functionally equivalent*, in the sense of Definition 4, to ant system,  $\mathcal{MAX-MIN}$  ant system, and ant colony system, respectively. Since the performance of the latter algorithms on the traveling salesman problem (as well as on many other problems) has been thoroughly studied in a number of previously published works, we refer the reader to the relevant literature such as, for example, Dorigo et al. [9], Dorigo and Gambardella [10], and Stützle and Hoos [11].

Nonetheless, another important issue needs to be addressed. The strong invariance is achieved in *siAS*, *siMMAS*, and *siACS* by normalizing the cost of each solution to be used for updating the pheromone. This normalization is obtained by dividing the cost of each solution by the cost of a reference solution. In this paper, we have adopted as a reference solution the one obtained by the nearest-neighbor heuristic. In principle, the normalization of solutions might have an impact on the speed of the algorithm and could therefore raise questions on the opportunity of adopting the strongly-invariant version of an ant colony optimization algorithm rather than the classical one.

The goal of this section is to show that the normalization has a negligible impact on the speed of the ant colony optimization algorithms considered in this paper. To this aim, we consider the ACOTSP program implemented by Thomas Stützle as a companion software for Dorigo and Stützle [2]. ACOTSP comprises the most widely adopted algorithms belonging to the ant colony optimization family, including ant system,  $\mathcal{MAX-MIN}$  ant system, and ant colony system. ACOTSP has been released in the public domain and it is available for free download.<sup>4</sup> Starting from the ACOTSP code, we obtained a straightforward implementation of *siAS*, *siMMAS*, and *siACS*, by modifying the heuristic information, the pheromone update rules, and the initialization, as defined in Sections 3, 4, and 5. Apart from these elements, the implementations of *siAS*, *siMMAS*, and *siACS* that we consider here are identical to the original implementations of ant system,  $\mathcal{MAX-MIN}$  ant system, and ant colony system, respectively. Since the difference between the strongly-invariant versions and their classical counterpart concerns only the heuristic information and the pheromone trail, no local search is considered in our experiments. The parameters of the algorithms are fixed as reported in Dorigo and Stützle [2]. See Table 1 for the details.

In our analysis, we consider 1000 instances of the traveling salesman problem generated through *portgen*, the instance generator adopted in the DIMACS TSP Challenge. More specifically, the instances we consider here consist each of 1000 integer-coordinate cities grouped in clusters and distributed in a square of size 10e+06.

<sup>4</sup><http://aco-metaheuristic.org/downloads/ACOTSP.V1.0.tar.gz>

Table 1: Values of the parameters adopted for the algorithms under analysis.

	<i>siAS</i> & AS	<i>siMMAS</i> & <i>MMAS</i>	<i>siACS</i> & ACS
$m$	100	100	100
$\rho$	0.5	0.02	0.1
$\alpha$	1	1	—
$\beta$	3	3	3
$q_0$	—	—	0.9

We measure the speed of the above implementations in terms of number of iterations per second. In order to obtain an accurate measurement of the speed, we run the implementations under analysis for 500 iterations, we discount the initialization time to obtain the number  $t$  of seconds needed for performing 500 iterations, and then we compute the speed by dividing 500 by  $t$ . The index we consider in our analysis is the *percent deviation* of the speed of the strongly-invariant version from the one of the corresponding original version. Formally, the percent deviation is defined as:

$$\%D_X = \frac{V_{siX} - V_X}{V_X} \times 100, \quad (10)$$

where  $V$  is the speed expressed in iterations per second, and  $X$  is the acronym of the specific algorithm under analysis, that is, AS for ant system, *MMAS* for *MAX-MIN* ant system, and ACS for ant colony system. For the sake of completeness, we mention here that all experiments have been run on a cluster of 33 units, each featuring two processors AMD Opteron™ 244, 2GB of memory, and running the Debian Sarge distribution of the Linux operating system.

The histograms in Figure 1(a) provide a representation of the observed percent deviation. For the three algorithms considered, the histograms are bell-shaped and reasonably symmetric. At a first visual analysis, the central tendency in the three histogram seems to be slightly to the left hand side of the origin which suggests, as *a priori* expected, that the strongly-invariant versions of the three algorithms are slightly slower than their corresponding original versions. Moreover, it can be observed that the three histogram are visibly leptokurtic. It can be therefore concluded that the distributions of the percent deviation depart from the normal one. This is confirmed by the graphs given in Figure 1(b) in which the quantiles of the percent deviation are plotted against those of the normal distribution.

After this preliminary visual analysis of the data, let us move to a quantitative analysis. In the following, we will present a number of statistical tests on the available data. All the tests are performed with a confidence level of 98%. We consider first a t-test [13] on the percent deviations for what concerns the three comparisons *siAS* vs. AS, *siMMAS* vs. *MMAS*, and *siACS* vs. ACS. In particular, we consider here a one-sided version since it can be *a priori* expected that the strongly-invariant implementation of each algorithm is slower than the corresponding original one. The results are reported in Table 2(a). As it can be observed, the mean values are very small in absolute value and, in the case of the comparison *siMMAS* vs. *MMAS*, the mean value is even in the positive. In none of the three cases the difference is significant. The last row of Table 2(a) reports a one-sided 98% confidence bound on the mean percent deviation: It can be stated with 98% confidence that *siAS* is not more than 0.29% slower than ant system. Similarly, at a confidence level of 98%, we can state that the speed of *siMMAS* and *siACS* are within 0.09% and 0.27% of the speed of *MMAS* and ACS, respectively.

As we have observed in our visual inspection of the data, and as it is confirmed by the results of the Shapiro-Wilk normality test [14] reported in Table 2(b), the observations appear to depart from the normal distribution. Indeed, this departure does not appear to be dramatic since the

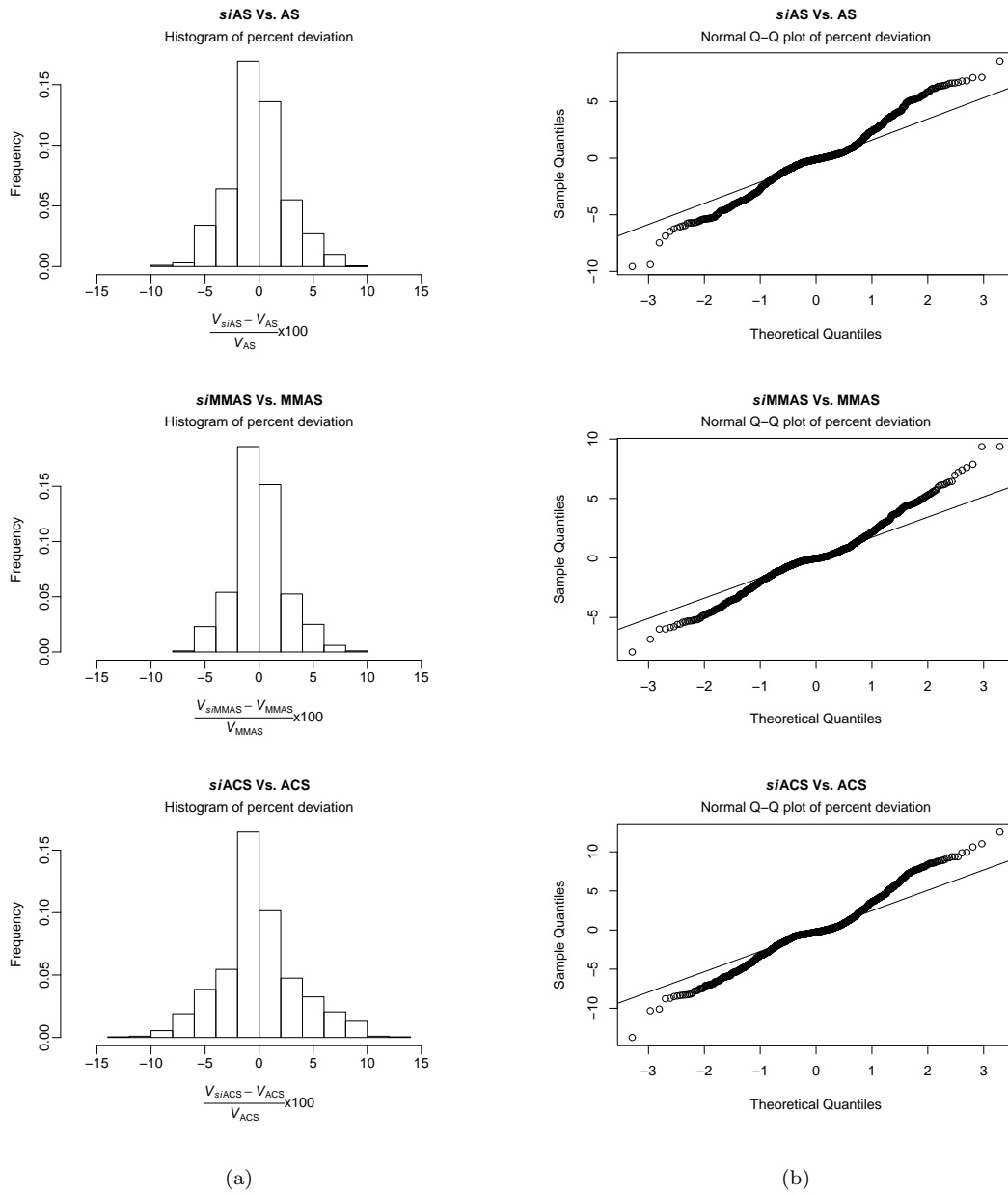


Figure 1: Percent deviation of the speed of the strongly-invariant version from the one of the original version of ant system, *MAX-MIN* ant system, and ant colony system. On the left hand side, in Figure 1(a), the histograms of the percent deviation are reported. On the right hand side, in Figure 1(b), the quantile-quantile plots against the normal distribution show that the distributions of the percent deviation significantly depart from the normal distribution itself.

Table 2: Statistical analysis of the percent deviation in speed of the strongly-invariant version from the original version of ant system,  $\mathcal{MAX}\text{-}\mathcal{MIN}$  ant system, and ant colony system. Both the t-test and the Wilcoxon test are one-sided with the alternative hypothesis that the percent deviation given in Equation 10 is negative, that is, the strongly-invariant version of the algorithm is slower than the classical one. The confidence level is 98%. The p-values typeset in bold are those that fall below the threshold of  $2.00\text{e-}02$ , under which the null hypothesis is rejected. In the tables, all figures concerning percent deviations are reported with two digits after the decimal point while all p-values are reported in scientific notation with 3 significant digits.

(a) Results of the t-test.

t-test	<i>siAS</i> vs. AS	<i>siMMAS</i> vs. $\mathcal{MMAS}$	<i>siACS</i> vs. ACS
mean	-0.12	0.07	-0.03
p-value	<b>7.37e-02</b>	8.18e-01	4.00e-01
98% conf. bound	-0.29	-0.09	-0.27

(b) Results of the Shapiro-Wilk normality test.

Shapiro-Wilk test	<i>siAS</i> vs. AS	<i>siMMAS</i> vs. $\mathcal{MMAS}$	<i>siACS</i> vs. ACS
p-value	<b>9.42e-11</b>	<b>1.52e-11</b>	<b>5.19e-12</b>

(c) Results of the Wilcoxon signed rank test.

Wilcoxon test	<i>siAS</i> vs. AS	<i>siMMAS</i> vs. $\mathcal{MMAS}$	<i>siACS</i> vs. ACS
median	-0.14	-0.00	-0.18
p-value	<b>1.35e-02</b>	4.98e-01	2.33e-02
98% conf. bound	-0.28	-0.12	-0.35

histograms reported in Figure 1(a) are nonetheless symmetric and unimodal. It is generally agreed that the t-test is robust against mild violations of the normality assumption and that, in case, it just loses power and fails gracefully. In any case, in order to avoid the risk that our analysis is invalidated by the non normality of the data and in order to fully convince our reader of the validity of the conclusions we draw, we resort to the Wilcoxon signed rank test [14]. This latter test does not rely on any hypothesis of normality of the data. Also in this case, we consider a one-sided version of the test and we work at a confidence level of 98%. The results of the Wilcoxon test are given in Table 2(c). They substantially confirm what already pointed out by the t-test with the only notable exception that in this case the difference between *siAS* and AS is statistically significant, albeit extremely tiny in absolute value, that is, 0.14%. It should also be observed that the p-value of the comparison *siACS* vs. ACS is very small but still above the threshold of significance. Moreover, in this case, the median of the percent deviation in the comparison *siMMAS* vs.  $\mathcal{MMAS}$  is zero and is not anymore in the positive as in Table 2(a).<sup>5</sup> The 98% confidence bound are basically unchanged with respect to those obtained through the t-test. Anyway, the fact that the Wilcoxon test proved to be more powerful than the t-test being able to detect significance where the latter

<sup>5</sup>The number that we actually observed is negative but very small. Since in Table 2 all percent difference and the relative bounds are reported up to the second digits after the decimal point, this number appear as  $-0.00$ .

failed—that is, in the comparison *siAS* vs. *AS*—somehow indicates that the distributions of the percent deviation significantly departs from normality and that the t-test is possibly not the best choice. For drawing some conclusions on the experimental analysis proposed in this section, we will therefore refer to the results obtained through the Wilcoxon test.

We can conclude that for what concerns ant system, the strongly-invariant version is indeed slightly slower than its classical counterpart. Nonetheless, with a confidence of 98%, we can state that the difference in speed is less than 0.28%. On the contrary, in our experimental setting we were not able to gather sufficient evidence that *siMMAS* and *siACS* are slower than the original *MAX-MIN* ant system and ant colony system, respectively: Although the medians of the percent deviations are slightly in the negative, we are not able to reject the null hypothesis with the confidence of 98% that we had fixed prior to running the experiments. In any case, should *siMMAS* be actually slower than *MAX-MIN* ant system, we could nonetheless state, with a confidence of 98%, that the difference in speed is less than 0.12%. Similarly, should *siACS* be slower than ant colony system, we could state, with a confidence of 98%, that the difference in speed is less than 0.35%.

Definitely, for ant system, *MAX-MIN* ant system, and ant colony system, the tiny difference in speed between the original version and the strongly-invariant version cannot be adopted as an argument against the latter.

## 7 Conclusions

Contrary to what previously believed [1], ant colony optimization appears to be invariant to the rescaling of problem instances. In this paper, we have formally proved that the three most successful and most widely adopted algorithms belonging to the ant colony optimization family—namely, ant system, *MAX-MIN* ant system, and ant colony system—are indeed *weekly-invariant*. In other words, the sequence of solutions they produce does not depend on the scale of the problem instance at hand. The proofs are provided with reference to the traveling salesman problem and are then extended to a generic problem. The technique adopted for proving the theorems is basically the same for the three algorithms. In the three cases, the proof is of inductive nature: We prove that if some conditions are fulfilled at the beginning of iteration  $h$ , then the solutions produced at iteration  $h$ , while solving any two linearly-related instances, are the same. Moreover, the same above conditions hold also at the following iteration  $h + 1$ . The prove is concluded by showing that the conditions are fulfilled at the beginning of the first iteration. This same proving technique can be adopted for formally showing the invariance of other algorithms belonging to the ant colony optimization family.

Moreover, the paper introduces three algorithms: *siAS*, *siMMAS*, and *siACS*. These algorithms are *functionally equivalent* to ant system, *MAX-MIN* ant system, and ant colony system, respectively, but they enjoy the further property of being *strongly invariant*. In other words, beside producing the same sequence of solutions irrespectively of any linear rescaling of the problem instance, they are such that the *pheromone* and the *heuristic information* do not change with the scale of the problem instance.

The attention on what in this paper we call *strong invariance* has been explicitly brought for the first time by Blum and Dorigo [1]. This property is definitely desirable for at least two main reasons: First, it reduces possible numerical problems in the implementations and contributes therefore to enhance the stability of the algorithm. Second, it greatly improves the readability of the solution process and provides the researcher, as well as the practitioner, with an important tool for analyzing the behavior of the algorithm. In order to achieve the strong invariance, Blum and Dorigo [1] have defined a new framework they named *hyper-cube*. An *hyper-cube* version of ant system, *MAX-MIN* ant system, or ant colony system is effectively a new algorithm which

shares with its originating (non-*hyper-cube*) version much of the underlying ideas but that is not functionally equivalent to the latter. The main advantage of the strongly-invariant algorithms we have proposed in the paper is indeed that they are proved to be functionally equivalent to their respective original counterpart. The properties of these algorithms do not need therefore to be studied from scratch: The results reported in the whole existing literature on ant colony optimization, which starts nowadays to be rather substantial, directly apply to these new algorithms. In particular, ant system,  $\mathcal{MAX-MIN}$  ant system, and ant colony system have been successfully applied to a variety of problems which means that we have for free an assessment of the performance of *siAS*, *siMMAS*, and *siACS* under a large number of experimental conditions.

By itself, the theorems proposed in Sections 3, 4, and 5, would not be sufficient for justifying the extension to the strongly-invariant algorithms of the results previously obtained on their classical counterparts. The full justification comes only from the combination of the above theorems with the empirical analysis given in Section 6. Indeed, one could suspect that, although the original and the strong-invariant versions obtain the same solutions, the strong-invariant could be slower since some extra computation is needed for normalizing the solutions to be used for pheromone update. The empirical analysis proposed in Section 6 shows that this is not the case: With 98% confidence, we can state that the difference in speed of *siAS* vs. *AS*, *siMMAS* vs. *MMAS*, and *siACS* vs. *ACS*, is less than 0.28%, 0.12%, and 0.35%, respectively. The differences are so tiny that they do not justify any concern about the relative performance of the strongly-invariant versions of the algorithms under analysis with respect to the classical versions.

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