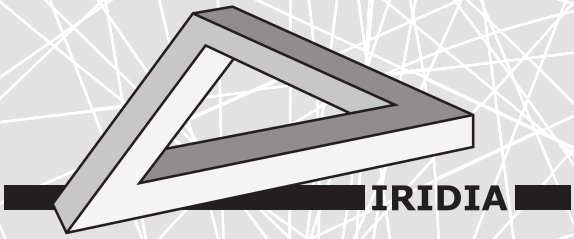


Université Libre de Bruxelles

Institut de Recherches Interdisciplinaires
et de Développements en Intelligence Artificielle

The Swarm/Potential Model: Modeling Robotics Swarms with Measure-valued Recursions Associated to Random Finite Sets

M. Pace, M. Birattari, and M. Dorigo

IRIDIA – Technical Report Series

Technical Report No.
TR/IRIDIA/2013-008

May 2013

IRIDIA – Technical Report Series
ISSN 1781-3794

Published by:
IRIDIA, *Institut de Recherches Interdisciplinaires et de Développements en Intelligence Artificielle*
Université Libre de Bruxelles
Av F. D. Roosevelt 50, CP 194/6
1050 Bruxelles, Belgium

Technical report number TR/IRIDIA/2013-008

The Swarm/Potential Model: Modeling Robotics Swarms with Measure-valued Recursions Associated to Random Finite Sets

Michele Pace, Mauro Birattari, *Senior Member, IEEE,* and Marco Dorigo, *Fellow, IEEE*

***Abstract*—We propose the S/P model (swarm/potential model) to describe the dynamics of robot swarms—that is, large groups of robots that coordinate without the use of any form of central control. The S/P model is a set of measure-valued equations that model the temporal evolution of a robot swarm and of a potential function that the swarm uses to self-coordinate. The S/P model is based on random finite sets (RFS), a general formalism for modeling multi-object systems. In this paper, we illustrate the S/P model by using it to describe and analyse three swarm robotics systems; we use particle approximations for the numerical solution of the model.**

I. Introduction

We propose a model to describe the dynamics of large groups of robots that operate in a self-organized way. Self-organization is the core concept in swarm robotics, an approach to collective robotics that seeks robustness and fault-tolerance via redundancy, local sensing, local communication, and the lack of any form of central control. Results have shown that, on the basis of local information gathered from the environment or received from neighbouring peers, robots in a swarm can make coherent decisions on tasks to be performed and on their respective priorities [1], [2]. Designing and analyzing robot swarms has proven to be a challenging task due to the complex relation occurring between the local (or *microscopic*) robot-robot and robot-environment interactions and the resulting collective (or *macroscopic*) behavior of the entire swarm. In order to design and analyze a robot swarm, it is important to find an appropriate level of description that, on the one hand, represents the relevant microscopic information that is needed to grasp the macroscopic dynamics and, on the other hand, is not flooded with excessive microscopic details. In this paper, we propose to use the random finite set (RFS) formalism [3], [4], [5] for modeling robot swarms. In particular, we illustrate a set of measure-valued equations that we call the S/P model—swarm/potential model. The S/P model describes the temporal evolution of a probabilistic description of a robot swarm and of a potential function that the robots use to encode and to share information that is relevant to achieve self-coordination. The premise is that the state of the robot swarm (i.e., the collection of the states of the individual robots) may be formulated as a random finite set, that is, as a random set in which the number of elements as well as their values are stochastic. The central thesis of the article is that this formulation leads to a natural and elegant description of large groups of robots and opens the door to efficient simulation algorithms. In order to facilitate the intuition behind the rather abstract measure-valued equations that we introduce, we maintain a strong emphasis on the spatiality of the models, that is, the underlying physical system is regarded as a discrete set of points, linked to the physical positions of the robots. Nothing, however, prevents the definition of RFS models for extended state spaces, such as the combination of the physical coordinates, speed, and an internal state of the robots.

To illustrate the S/P model, we present three case studies. The first two concern wheeled robots that move in closed environments and exhibit two collective behaviors commonly studied in swarm robotics: movement in the direction of a light [6] and aggregation [7], [8]. The last case study concerns the so-called *chemical robots* [9]: colloidal particles with diameters of about 200 nm or less, that are intended to be injected in the human body to accomplish tasks such as chemical sensing [10], targeted drug delivery and micro-assembly [11].

The rest of the paper is organized as follows. In Section II we provide a short list of references to relevant works on modeling in swarm robotics and pointers to relevant literature on random finite sets. In Section III we summarize the concepts of the random finite set framework that will be used throughout the rest of the paper. In Section IV we introduce the general S/P model (swarm/potential model). In Section V we specialize the S/P model for swarm robotics. In Section VI we develop an approximate solution of the S/P model. In Section VII we illustrate the S/P model on the three case studies. We conclude the paper in Section VIII.

II. Related works

In order to understand and model the relationship between the local and global aspects of swarm systems (the so called *micro-macro link*), several approaches have been proposed. Early works investigate the derivation of coupled differential equations to describe the average state of the swarm in time [12]. When this description is able to capture the relevant characteristics of the system, a large corpus of well established theoretical techniques can be used to analyze the stability and long term behavior of the macroscopic models. These techniques have been used to study swarming and social aggregation [7], collective decision-making [13], [14] and foraging [15]. More recent approaches to model robotics swarms use

The authors are with IRIDIA, Université Libre de Bruxelles, Belgium (e-mail: Michele.Pace@ulb.ac.be; mbiro@ulb.ac.be; mdorigo@ulb.ac.be)

analogies from the domain of chemical modeling [16] and methods from statistical physics [6], [17]. In particular, in [6] macroscopic models based on Fokker-Planck equations are derived from microscopic models.

Differently, in this paper the main idea is to model the objects at each time step as RFSs, that is *set-valued random variables*, and then to characterise the uncertainty about the collective motion of the swarm by propagating in time the intensity of the associated point process [18], [5], [4], [3]. Random set theory was first systematically examined in connection with statistical geometry by Kendall [18] and Matheron [3] in the mid-1970s and then applied to two-dimensional image analysis by Serra [19]. Since then, it has become a basic tool in theoretical statistics and it has generated a substantial amount of innovation in the fields related to data fusion and multi-target tracking [4], [20], [21], [22]. An early application to robotics has been investigated in [23].

III. The Random Finite Set Framework

A random finite set can be described as a finite, set-valued random variable where not only the elements but also the cardinality of the set is random [24].[1] The realization of a RFS is thus an unordered and finite set of elements distributed according to a common probability distribution. This concept is particularly relevant in the field of swarm robotics, where the objective is normally not to describe the state of each single robot, but rather to obtain a descriptive characterization of the swarm as a whole. This section introduces the RFS formalism that will be used throughout the article. The measure-valued recursion describing the evolution in time of the intensity of a Poisson RFS associated to the macroscopic behavior of robotic swarms is introduced in Sec. IV.

The set of all finite positive measures on some measurable space E will be denoted by $\mathcal{M}(E)$, the subset of all probability measures on E by $\mathcal{P}(E) \subset \mathcal{M}(E)$ and the Banach space of all bounded and measurable real-valued functions on E by $\mathcal{B}(E)$. Let $\mu(f) \stackrel{\text{def}}{=} \int f(x)\mu(dx)$ be the Lebesgue integral of a function f with respect to a measure $\mu \in \mathcal{M}(E)$. A bounded positive integral operator Q from a measurable space E_1 into a measurable space E_2 is an operator $Q : f \in \mathcal{B}(E_2) \mapsto Q(f) \in \mathcal{B}(E_1)$ such that the functions

$$x \mapsto Q(f)(x) \stackrel{\text{def}}{=} \int_{E_2} Q(x, dy) f(y)$$

are measurable and bounded for some measure $Q(x, \cdot) \in \mathcal{M}(E_2)$. These operators induce a dual operator $(\mu Q)(dy) \stackrel{\text{def}}{=} \int \mu(dx) Q(x, dy)$ from $\mathcal{M}(E_1)$ into $\mathcal{M}(E_2)$ which will be used to model the way in which a probabilistic descriptor associated to a swarm of robots changes in time. Intuitively, the probability (or intensity) mass that characterizes the state of the system is *transported* in time according to forces and dynamics that are captured by the analytical form of the *transport* operator Q.

[1]Note that the terms *random finite set* and *simple-finite point process* are normally used to refer to the same mathematical object.

Let $g : x \in E \mapsto g(x) \in]0, +\infty[$ be a bounded positive function. A measure $\eta \in \mathcal{M}(E)$ can be transformed into a probability measure by using the Boltzmann-Gibbs transformation $\Psi_g : \eta \in \mathcal{M}(E) \mapsto \Psi_g(\eta) \in \mathcal{P}(E)$ defined in [25], Pag. 20, as:

$$\Psi_g(\eta)(dx) \stackrel{\text{def}}{=} \frac{1}{\eta(g)} \, g(x) \, \eta(dx) \tag{1}$$

provided $\eta(g) > 0$. One of the advantages that come from the use of the Boltzmann-Gibbs transformation is that its particle approximation is obtained straightforwardly by resampling the particles approximating the original measure η according to their weights obtained by the evaluation of the potential function g. An introduction to particle methods with a detailed explanation of the Boltzmann-Gibbs transformation is given in [25]. Additional references covering the fundamentals of the interacting-particle systems applied to filtering problems can be found in [26] and [27].

The first moment measure M of a RFS X is the analogue of the expectation and it is commonly called *intensity measure*. It is defined for any subset S of the space E, $S \subseteq E$, by

$$M(S) = \mathbb{E}\left[|S|\right] \stackrel{\text{def}}{=} \int_S \gamma(x) dx \tag{2}$$

where $|S|$ denotes the number of points in the subset S. $M(S)$ gives the expected number of points of X contained in S. The last equality holds if the measure M admits a density $\gamma : E \to [0, +\infty[$. In point process theory γ is called *intensity function* and is simply referred to as the intensity.

Just as the density of a continuous random vector represents the zero-probability event of a particular realization of the random vector, the intensity $\gamma(x)$ represents the zero-probability event $\mathbb{P}(x \in X)$. The intensity $\gamma(x)$ is usually multimodal and the peaks are the regions of high object intensity.

In the following, we focus on a specific class of random finite sets that have the important property to be uniquely characterized by their intensity function: the Poisson RFS. In this case, the integral of the intensity over the domain (sometimes called mass) $\gamma(f) \stackrel{\text{def}}{=} \int \gamma(x) f(x) dx$ when $f(x) = 1$ corresponds to the expected number of points, and it will be denoted by $\gamma(1) \geq 0$.

A realization of a Poisson point process consists of N points i.i.d. with distribution $\eta(dx) = \gamma(dx)/\gamma(1)$ where N is an integer-valued Poisson random variable with parameter $\gamma(1)$. Essentially, the Poisson RFS characterizes a set of points with no interaction, that is, complete spatial randomness. In the context of swarm robotics, this property implies that the state of a robot and its dynamics are independent of the state of the other robots. While this may not be true in the general case, it can be considered as an acceptable approximation for systems where robots have a low level of mutual interdependence.

IV. The S/P Model

This section introduces a coupled system describing the propagation in time of the intensity function γ_t associated to a Poisson point process that we use to describe the state of a swarm at time t. The use of the intensity of the point

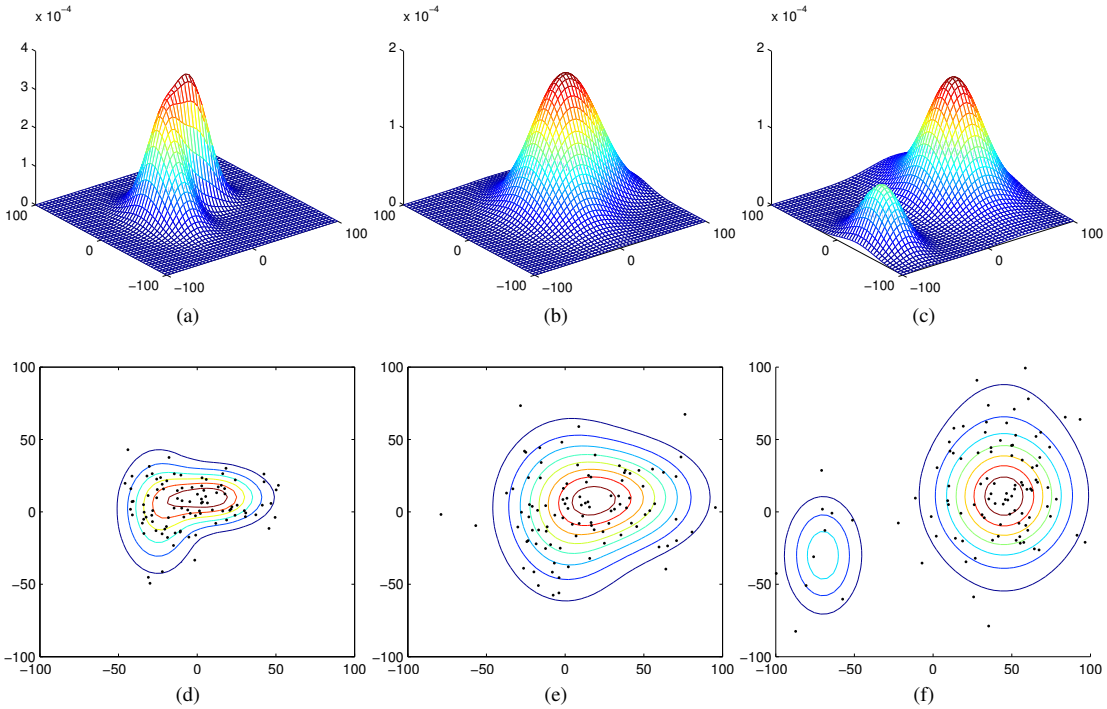


Fig. 1: Intensity functions $\gamma_t, \gamma_{t+1}, \gamma_{t+2}$ of the point process associated to the robotics swarm at time $t, t+1$ and $t+2$ (a),(b),(c) and realizations of the point process (d),(e),(f)

process as a probabilistic descriptor allows for the characterization of the uncertainty about the number and distribution of the robots while keeping the computational complexity significantly lower than other probabilistic descriptors, such as the full multi-object probability density [3], [21]. Figures 1(a-b-c) provide an intuitive illustration of the probabilistic description of a swarm's state (in the example the state is given by the 2-dimensional spatial position) in time as a sequence of intensity functions $\gamma_t, \gamma_{t+1}, \gamma_{t+2}$. Figures 1(d-e-f) show a realization of the point process X_t, X_{t+1}, X_{t+2} superimposed to the profile of the intensity functions. The points represent one of the infinite possible realizations of a Poisson RFS characterized by the respective intensities, corresponding to a possible state configuration of the robots. The objective is to couple the equation describing the evolution of the intensity with an equation capturing the interaction of the swarm with the environment. The dynamics of the interactions will be modeled with a positive function which will be referred to as *potential*. Phenomena of self-interaction and feedback, for instance, are captured by modeling the way robots modify the potential function and how, in turn, the potential modifies the behavior of the robots. The system we are considering, referred to in the following as the S/P model, has the following form:

$$\begin{cases} \gamma_{t+1} & = \gamma_t Q^{g_t}_{t+1,\gamma_t} + \mu_{t+1} \\ g_{t+1} & = \Phi(g_t, \gamma_t) \end{cases} \tag{3}$$

where the propagation of the intensity depends on the potential function g_t (the dependency is made explicit by using the superscript g_t in Q) while the potential, in turn, depends on the intensity function γ_t. This dependency captures the fact that robots may be able to modify the environment with their actions. The evolution in time of the expected number of robot is described both via the term $Q^{g_t}_{t+1,\gamma_t}$ and μ_{t+1}. The mass of $Q^{g_t}_{t+1,\gamma_t}$ models modifications of the expected number of robots in a region of space that are function of the intensity γ_t, that is of the expected number of robots currently present in the region. This capture the phenomenon of spawning and the failure of robots. On the other hand, μ_{t+1} is the intensity of a Poisson point process modeling new robots coming in the environment independently of the current distribution of robots. This captures the deployment of further robots.

The S/P model of equation (3), inspired from a model discussed in [28], takes into account the effects of the interactions between the robots and a potential representing a set of characteristics of the environment that can be perceived by the robots. Dynamics of the potential that do not depend on the robots, such as potential diffusion and dissipation, are also modeled in the operator Φ (via the dependence of g_{t+1} on g_t). We will illustrate the S/P model by using it to describe and analyse three swarm robotics scenarios: movement in the direction of a light [6], swarm aggregation [8], [29], and coordinated motion of colloidal particles (so-called *chemical robots* [30]). The actual modeling of specific swarm robotics systems is obtained by appropriately defining the operator $Q^{g_t}_{t+1,\gamma_t}$, which governs the evolution of the intensity in time, and the operator Φ, which governs the evolution of the potential in time. In addition, robots that are added to the swarm are modeled by μ_{t+1}.

In this framework, robots are modeled as memory-less. This constraint, introduced to make the equations more tractable, is acceptable when robots make limited use of their memory and do not take decisions based on stimuli or events happened in the distant past. Consequently, typical characteristics of swarm robotics such as sharing of information or positive and negative feedback dynamics have to be modeled in the potential via the operator Φ.

The numerical solution of the S/P model can be obtained by adopting a stochastic particle interpretation for which many stability properties as well as a series of estimators for the approximation errors are available [31].

V. Specializing the S/P Model for Swarm Robotics

In this section we introduce a specialization of the S/P model (Eq. (3)) that makes it suitable for swarm robotics. To do so, we define the form of the operator $Q^{g_t}_{t+1,\gamma_t}$ associated to the evolution of the intensity and the operator Φ associated to the evolution of the potential.

In the context of swarm robotics, this specialization corresponds to the microscopic model that describes the way robots sense the environment (i.e., how robots measure the potential function in their surroundings), how they react by choosing from a set of possible actions, and how they modify the potential function.

Section V-A introduces a model for the dynamics of the robots, while section V-B describes the evolution of the potentials.

A. Robots dynamics

The first step towards the specialization of the S/P model is to characterize the probability with which a robot in state $X_t = x$ moves to an infinitesimal region dy centered in $X_{t+1} = y$. We assume that this transition, given by the normalized version of $Q^{g_t}_{t+1,\gamma_t}$, is time-homogeneous and independent of γ_t. The transport operator can therefore be written as $Q^{g_t}(x, dy)$. The removal of the dependency on time indicates that the robot behaviour does not change in time. The removal of the dependency on the intensity γ_t indicates that the robot behavior does not depend on the overall state of the swarm. These assumptions allow for a simplification of the S/P model equations. Additionally, we assume that $Q^{g_t}(x, \cdot) \in \mathcal{P}(E)$ for each $x \in E$ (this is equivalent to say that the transport operator Q^{g_t} has integral 1). This implies that the number of robots in the scene is expected to remain the same. We model the dependence of the dynamics of the robots on the value of the potential function by the Markov operator originally proposed in [32]:

$$Q^{g_t}(x, dy) = Q_0(x, dy)e^{-\lambda(g_t(x)-g_t(y))^+} + Q_1(x, dy)\left(1 - \int Q_0(x, dy)e^{-\lambda(g_t(x)-g_t(y))^+}\right) \quad (4)$$

This operator captures the intrinsic dynamics of the robots (modeled by $Q_0(x, dy)$) as well as the effect of the potential field. The parameter λ specifies the sensitivity of the robots to the difference in potential between the current state $X_t = x$ and one of the possible future states X_{t+1}.

Suppose a robot is in state $X_t = x$ at time t, possibly representing its physical location. Its future state X_{t+1} at time $t + 1$ depends on the environmental conditions around it, captured by the potential function, and on the robotics controller. In Eq. (4) the operator $Q_0(x, dy)$ models the dynamics of the robot in absence of any external conditioning, that is, when $g_t(\cdot)$ is constant. When $g_t(\cdot)$ is not constant, the robot dynamics is influenced by the potential. The probability that $X_{t+1} \in dy$ is $e^{-\lambda(g_t(x)-g_t(y))^+}$: that is, it depends on the difference of potential between the current and the new state. If the movement is not performed, the new state of the robot if sampled according to the operator $Q_1(x, dy)$. This happens with probability $(1 - \int e^{-\lambda(g(x)-g(y))^+})$. It is important to note that even though the dynamics of the robots given by Eq. (4) are completely specified by the transition probabilities $Q_0(x, dy)$, $Q_1(x, dy)$ and by the parameter λ, no constraints are imposed on the development of the algorithm used by the robots to take decisions. In practical terms, the link between the analytical form of the operator $Q^{g_t}_{t+1,\gamma_t}$ and the implementation of robotics controllers can be established in two ways. If the S/P model (with $Q^{g_t}_{t+1,\gamma_t}$ in its simplified version as in Eq. (4)) is used to derive specific behaviors to be implemented on the robots, the form of Q_0, Q_1, and λ are first investigated numerically. When the analytical form of the operators that generate the desired swarm-level behavior is found, the robotic controller can be implemented as a series of actions that individual robots take with certain probabilities. If, however, a robotic controller is already available (for example implemented by using other methodologies [33]), one must first consider if Eq. (4) is sufficiently general to model the behavior of the robots. If this is the case, the form of $Q_0(x, dy)$, $Q_1(x, dy)$ and the value of the parameter λ that capture the dynamics of the robot must be derived. Once the characteristics of the robot controller are satisfactorily captured by Q^{g_t}, the system can be studied numerically to gain insights on the swarm-level dynamics. Otherwise, if Eq. (4) is not sufficiently general to model the behavior of the robots, an alternative and possibly more complex expression of $Q^{g_t}_{t+1,\gamma_t}$ is necessary.

The advantage of adopting Eq. (4) as a probabilistic description of the microscopic dynamics of the robots is that if closed form solutions are not available, as it is generally the case, generating samples from $Q^{g_t}(x, \cdot)$ is straightforward and can be done with the following procedure:

- Draw a sample $y_0 \sim Q_0(x, \cdot)$ and a sample $Y_1 \sim Q_1(x, \cdot)$
- If $g(y_0) \geq g(y_1)$ accept y_0 and do the transition $x \to y_0$
- If $g(y_0) \leq g(y_1)$ accept y_0 with a probability $e^{-\lambda(g(x)-g(y))^+}$ and do the transition $x \to y_0$ otherwise do the transition $x \to y_1$

Figure 2 shows the acceptance probability of the samples obtained from Q_0 as a function of the difference of potential $(g(x) - g(y))$ when different values of the parameter λ are used. For low values of λ, robots accept the transitions towards regions with lower potential with relatively high probability. In this case, the gradient of the potential exerts only a weak

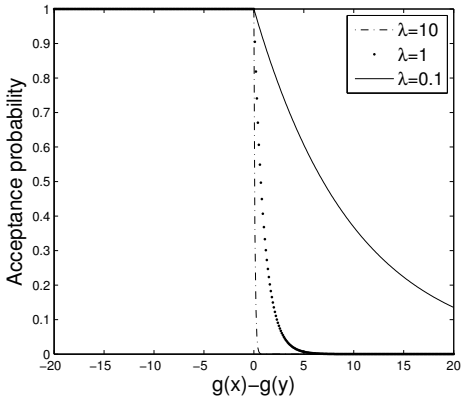


Fig. 2: Acceptance probability of the sample obtained from Q_0 with decreasing values of λ.

influence on the behavior of the robots. For high values of λ, on the contrary, the movement is accepted almost exclusively if the new location has a higher potential than the current location. The macroscopic result is that the swarm is strongly attracted towards regions of the space associated with high potentials.

To clarify this important point, Figure 3 illustrates the evolution of the probability density (approximated with 1000 particles) of the position $x = [\psi_1, \psi_2]^T$ of the robots moving according to Eq. (4) with $\lambda = 50$, $Q_0(x, \cdot) = \mathcal{N}(\cdot; x, 0.1\mathbf{I}_2)$, $Q_1(x, \cdot) = \mathcal{N}(\cdot; x, 10^{-4}\mathbf{I}_2)$. The n-dimensional identity matrix is denoted by $\mathbf{I}_n$. The value of the potential is represented as a grayscale image (white associated to higher potentials). In time, the robots are attracted by high potential regions. The particle approximation provides, at each time step, a characterization of the probability of finding a certain number of robots in a certain region of the space.

B. Potential dynamics

The second step to complete the specialization of the S/P model is to characterize the form of the recursion of the potential in Eq. (3). In doing so, we focus on two types of intrinsic potential dynamics, **dissipation** and **diffusion**, and on potential dynamics due to the actions of the robots, **stigmergy**[2].

In order to avoid negative potentials, we assume that the potential π_t is given in terms of positive measures: $\pi_t : \mathcal{S} \mapsto \mathbb{R}$, where $\mathcal{S}$ is the set of subsets of E. According to the notation previously introduced, the integral of the potential over the state space is denoted by $\pi_t(1) \stackrel{\text{def}}{=} \int_E \pi_t(x)dx$. Simililarly to what discussed in Sec. V of [37], the evolution of the potential

[2]Stigmergy is a mechanism of indirect coordination observable in insect societies such as ant colonies. It is based on the ability of the individuals to create dissipative fields by spreading chemical substances into the environment. Other individuals then sense the chemical fields and are influenced to perform certain activities or to assume certain behaviors. The spreading of additional chemical substances as a reflection of these activities tends to create reinforcement dynamics. Stigmergy allows complex social behaviors to emerge from the actions of the individuals without the need for central planning, control, or even direct communication [34], [35], [36].

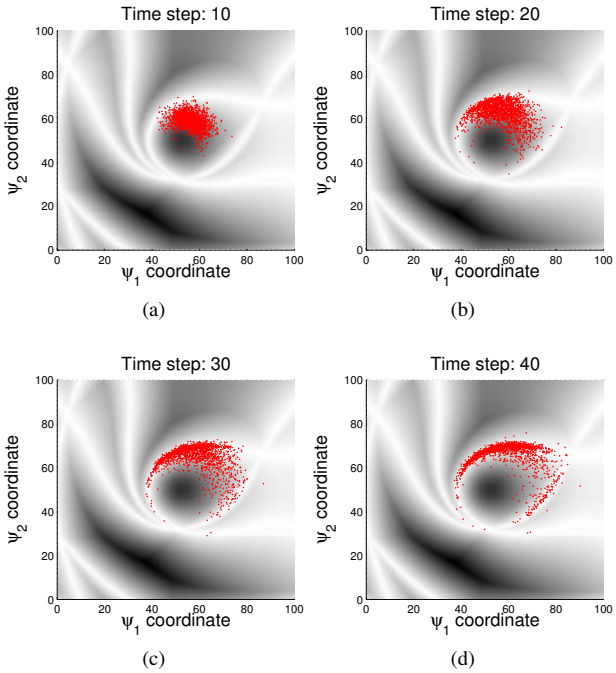


Fig. 3: Distribution of the position of the swarm at time $t = 10$ (a), $t = 20$ (b), $t = 30$ (c), $t = 40$ (d). In the example: $\lambda = 50$, $Q_0(x, \cdot) = \mathcal{N}(\cdot; x, 0.1\mathbf{I}_2)$, $Q_1(x, \cdot) = \mathcal{N}(\cdot; x, 10^{-4}\mathbf{I}_2)$, where $\mathbf{I}_2$ denotes the identity matrix with dimension 2. Low potential regions are shown in gray/black.

is modeled as follows:

$$\pi_{t+1} = \pi_t R + C_{\pi_t, \gamma_t} \tag{5}$$

or, equivalently:

$$\pi_{t+1}(dy) = \int \pi_t(x) R(x, dy) dx + C_{\pi_t, \gamma_t}(dy) \tag{6}$$

where C_{π_t, γ_t} denotes a positive measure on $\mathcal{S}$ modeling the robots' stigmergic contribution to the potential. The subscripts π_t and γ_t are used to stress the fact that, at any given time, the increase of the potential occurs in a way that depends on the intensity of the robots at time t (represented by γ_t), and possibly on the potential function π_t itself. The operator capturing the diffusive and dissipative dynamics of the potential is denoted by $R(x, dy)$.

C. S/P Model dynamics

The S/P model describing the evolution of the potential and particle approximation of the intensity measure is as follows:

$$\begin{cases} \gamma^N_{t+1} & = \gamma^N_t Q^{\pi_t} + \mu^N_{t+1} \\ \pi_{t+1} & = \pi_t R + C_{\pi_t, \gamma_t} \end{cases} \tag{7}$$

where N is used to stress the fact that we are using N particles to approximate the flow of γ_t. The limit $N \to \infty$ formally corresponds to the S/P Model of Eq. (3):

$$\begin{cases} \gamma_{t+1} & = \gamma_t Q^{\pi_t} + \mu_{t+1} \\ \pi_{t+1} & = \pi_t R + C_{\pi_t, \gamma_t} \end{cases} \tag{8}$$

As stated in Eqs. (7) and (8), the expression of the potential function is assumed to be known in closed form at each time step. No particle approximation of π_t is thus required. This is true only for a limited set of models. The general case would require an approximation of the potential function as well: $\pi^N_{t+1} = \pi^N_t R + C_{\pi^N_t, \gamma^N_t}$. For the sake of clarity and ease of implementation, we assume that an analytical form of π_t is available at each time step. Section VI-B discusses the implications of this assumption.
Figure 4 illustrates the evolution in time of the measures associated to the system (8). Arrows are used to show the dependencies between the elements of the recursion.

VI. Solution of the Specialized S/P Model

In this section we discuss the numerical techniques used in the rest of the article to approximate in time the solution of the specialized S/P model of Eq. (7). In particular, Sec.VI-A introduces the theoretical background of the particle approximation techniques used to approximate the first equation of the S/P model. Sec. VI-B discusses a Gaussian form of the potential function that allows for an efficient computational treatment of the second equation of the S/P model.

A. Mean field particle interpretation

The evolution of the intensity γ_t can be described by decoupling the process in the pair $(\gamma_t(1), \eta_t) \in (\mathbb{R}_+ \times \mathcal{P}(E))$, where $\gamma_t(1)$ represents the mass at time t and the normalized intensity $\eta_t = \gamma_t/\gamma_t(1)$ is the density function at time t. A way to solve the non-linear equations of the S/P model is to find a judicious probabilistic interpretation of the flow of normalized intensity $\eta_t = \gamma_t/\gamma_t(1)$ and to devise a particle approximation scheme for the probability density η_t such that the total mass of the process at time $(t+1)$ can be computed in terms of the particle approximations. We use the so-called mean field-type interpretation that is based on the fact that the density η_{t+1} can be thought as a non-linear Markov chain whose elementary transitions depend both on the distributions η_t and on the mass process $\gamma_t(1)$.

More precisely, as shown in Fig. 3, the approximation of the normalized intensity measures η_t and process mass $\gamma_t(1)$ is obtained by using a population of particles that evolve according to specifically designed probabilities. These discrete approximations are denoted by:

$$\eta^N_t = \frac{1}{N}\sum_{i=1}^{N} \delta_{x^i_t} \quad \text{and} \quad \gamma^N_t = \gamma^N_t(1)\ \eta^N_t \tag{9}$$

where N denotes the number of particles used to approximate η_t, and $\delta_{x^i_t}$ denotes the Dirac delta, which can be interpreted as a point mass at $x^i_t \in E$. A detailed description of the origin of these particle approximations can be found in [31], [38] and reference therein.

As it has been discussed in [25], [26], [27], [31], the non-linear transformation of the normalized intensity η_t can be rewritten in the form of the product of a selection operator S_{t,γ_t} and an update operator M^g_{t+1,γ_t}, both depending on the measure γ_t:

$$\eta_{t+1} = \eta_t S_{t,\gamma_t} M^g_{t+1,\gamma_t} \tag{10}$$

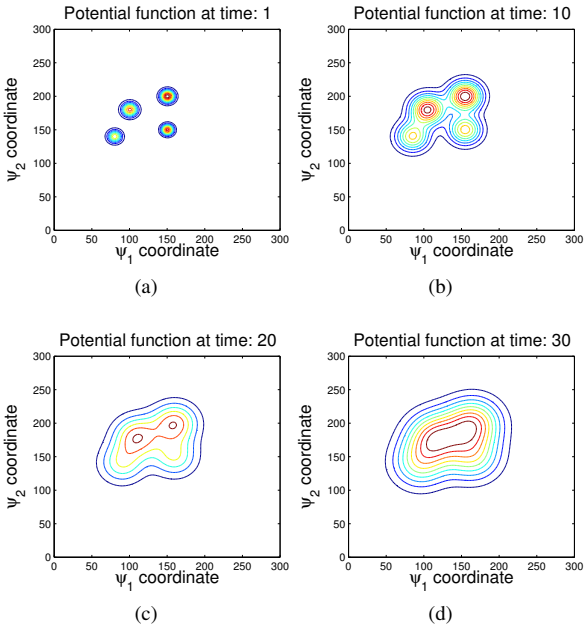


Fig. 5: Diffusion and dissipation of the potential corresponding to the solution of equation (13) in the Gaussian case.

and $\eta_t S_{\eta_t}$ can be written in term of the Boltzmann-Gibbs transformation: $\eta_t S_{\eta_t} = \Psi_{g_{t,\gamma_t}}(\eta_t)$ [27], [38].

Therefore, the solution of the γ_t recursion in the S/P model, Eq. (3), is given by the set of particles evolving according to these selection and propagation mechanisms:

$$\begin{pmatrix} \gamma^N_t(1) \\ \eta^N_t \end{pmatrix} \xrightarrow{\text{selection}} \begin{pmatrix} \widehat{\gamma}^N_t(1) \\ \widehat{\eta}^N_t \end{pmatrix} \xrightarrow{\text{propagation}} \begin{pmatrix} \gamma^N_{t+1}(1) \\ \eta^N_{t+1} \end{pmatrix}$$

When a large number of particles are used, their distribution in time provides both a good approximation of the flow of intensity γ_t, as shown in Fig. 3, and of the mass terms $\gamma_t(1)$.

B. A Gaussian representation of the potential

The potential function π_t can be represented as a weighted mixture of Gaussian terms. In particular, we assume that the potential at each time step t can be written in the form:

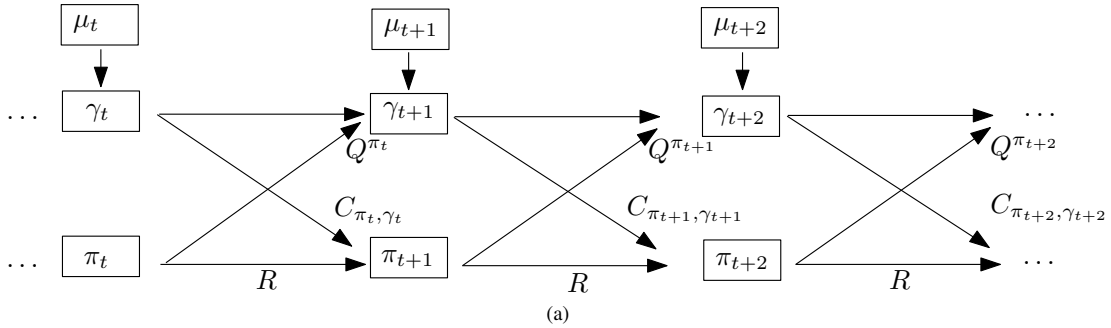
$$\pi_t(x) = \sum_{i=1}^{J_t} w^i_t \mathcal{N}(x; m^i_t, P^i_t) \tag{11}$$

where $\mathcal{N}(x; m, P)$ denotes a Gaussian term centered in m with covariance matrix P:

$$\mathcal{N}(x; m, P) = |2\pi P|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(x-m)^T P^{-1}(x-m)\right\} \tag{12}$$

w^i_t represents the weight of the i-th Gaussian in the mixture and J_t the total number of terms in the approximation of the potential at time t.

We assume that the dissipation of the potential is described by a constant coefficient $\alpha_t(x) = \alpha$ and that the diffusion of the potential and the contribution of the robots to the potential are linear-Gaussian. These assumptions are sufficient



(a)

Fig. 4: Evolution in time of the measures associated to the system (8). The dependencies between the potential and the intensity function is illustrated with arrows. Each arrow specifies the dependency of each term on other terms available during the previous time-step. The intensity γ_{t+1} associated to the state of the system at time $t+1$, for instance, depends on the potential π_t at time t as well as on γ_t and on the intensity on the robots appearing at time $t+1$. Similarly, the potential π_{t+1} at time $t+1$ depends on the intensity γ_t and on the potential π_t of the previous time step.

to characterize in closed form the potential π_t at each time step [39] [40]:

$$\pi_{t+1}(dy) = \underbrace{\alpha}_{\text{dissipation}} \int \pi_t(x) \underbrace{D(x,dy)}_{\text{diffusion}} dx \quad + \int \underbrace{C_{\pi_t,\gamma_t}(x,dy)}_{\text{contribution}} dx \quad (13)$$

where $R(x,dy) = \alpha\, D(x,dy)$ and the contribution to the potential $C_{\pi_t,\gamma_t}(x,dy)$ is a mixture of Gaussian terms. Equation (13) states that the potential in an infinitesimally small region dy at time $t+1$ is given by the sum of the potential that is transferred to dy from the other regions of the space (due to diffusive dynamics) and the additional potential released by the swarm in dy. The term α accounts for the dissipation, the term $D(x,dy)$ for the diffusion dynamics of the potential, and the term C_{π_t,γ_t} for the contribution of the swarm to the potential at time t. In this latter term, the subscripts π_t and γ_t stress that the contribution to the potential depends on the distribution of the robots and may as well depend on the value of the potential at time t.

To simplify the formulae, we make the reasonable assumption that:

$$C_{\pi_t,\gamma_t}(x,dy) = \gamma_t(x)\, C_{\pi_t}(x,dy)$$

By using the particle approximation of the intensity $\gamma_t(x)$ defined in Eq. (9), we can write:

$$\int C_{\pi_t,\gamma_t}(x,dy)\,dx \approx \frac{\gamma_t(1)}{N} \int \sum_{i=1}^{N} \delta_{x_t^i}\, C_{\pi_t}(x,dy)\,dx = \frac{\gamma_t(1)}{N} \sum_{i=1}^{N} C_{\pi_t}(x_t^i,dy) \quad (14)$$

Therefore, Eq. (13) can be written as:

$$\pi_{t+1}(dy) \approx \alpha \int \pi_t(x) D(x,dy) dx \quad + \frac{1}{N} \sum_{i=1}^{N} c_t^i\, \bar{C}_{\pi_t}(x_t^i,dy) \quad (15)$$

where

$$c_t^i = \gamma_t(1) \int C_{\pi_t}(x_t^i,dy)$$

and $\bar{C}_{\pi_t}(x_t^i,dy)$ is $C_{\pi_t}(x_t^i,dy)$ normalized by its integral.

The evolution in time of the potential, Eq. (13), has the following Gaussian form: at time t it is represented by the Gaussian mixture of Eq. (11), while at time $t+1$ it is given by

$$\pi_{t+1}(x) = \sum_{i=1}^{J_t} w_{t+1}^i\, \mathcal{N}(x; m_{t+1}^i, P_{t+1}^i) \quad + \sum_{i=J_t+1}^{J_t+1+N} w_{t+1}^i\, \mathcal{N}(x; m_{t+1}^i, P_{t+1}^i) \quad (16)$$

where $w_{t+1}^i = \alpha\, w_t^i$, for $i = 1,\ldots,J_t$; $w_{t+1}^i = c_t^j/N$, for $i = J_t+1,\ldots,J_t+1+N$ and $j = i - J_t + 1$; m_{t+1}^i and P_{t+1}^i, for $i = 1,\ldots,J_t$, are means and covariance matrices of the Gaussians resulting from the product between the terms of $\pi_t(x)$ and $D(x,dy)$, calculated by using the well known closed form solutions described in [39] and [40]; $m_{t+1}^i = x_t^i$, for $i = J_t+1,\ldots,J_t+1+N$; and P_{t+1}^i, for $i = J_t+1,\ldots,J_t+1+N$ is a design parameter representing the covariance matrix of the contribution of a robot to the potential function.

A comparison between Eq. (11) and Eq. (16) indicates that the number of Gaussian terms used to approximate π increases in time. In practical implementations of the S/P model, this problem can be addressed with a pruning operation: the weight w of each Gaussian term decreases exponentially in time due to dissipation; when the weight of a Gaussian term reaches a predefined threshold, the term can be neglected and is not considered in further computation.

VII. Case studies

In this section, we develop the concepts introduced in the previous sections for a series of swarm robotics scenarios: movement in the direction of a light, swarm aggregation, and coordinated motion of colloidal particles. We define the operators of the specialized S/P model so that its solution captures classical behaviors encountered in swarm robotics. As

previously noted, the model is composed of two equations: one describes the dynamics of the robots and the other describes the dynamics of the potential function. These two equations are coupled: the movement of the robots depends on the potential function via the transport operator; and the evolution of the potential function depends on the distribution of the robots.

In the first example, we focus on the first equation of the specialised S/P model of Eq. (7) and we show how it describes the collective dynamics of the swarm starting from the behavior of the individual robots. We also derive a particle approximation of the solution. In the second example, we focus on the second equation and we discuss the potential function that influences the behavior of the robots. Finally, in the third example we address a more complex scenario from the recently introduced field of chemical robots [30]: we use the S/P model to describe the cohesive motion of chemical robots along a preferential direction. These chemical robots coordinate by modifying the concentration of a chemical in the environment.

A. Swarm motion towards a light

In this section, Eq. (7) is solved to show that the S/P model can describe the motion of a robot swarm towards a light. Collective motion towards a light is a classical behavior studied in swarm robotics that has been discussed, for example, in [41] and [6].

In order to illustrate the mean-field approximation of the flow of the intensity measures, we focus here exclusively on the first equation of the S/P model: robots do not modify the environment, hence the second equation is set to zero. The appearance of new robots at time $t+1$ is modeled as a RFS with a known intensity μ_t. The number of robots appearing at each time step is described by a Poisson distrution with mean $\mu_t(1)$, and their initial position has probability density $\mu_t/\mu_t(1)$. Robots are equipped with a light sensor that allows them to estimate the direction of the light. Despite the fact that robots sense the direction of the light and deterministically attempt to move towards it, due to sensing and actuation errors their behavior appears to be stochastic and can be conveniently described as a random walk biased by a light field. The following Markov operator captures the main elements of the robots dynamics:

$$Q(x, dy) = \frac{l(y)M'(x, dy)}{\int l(y)M'(x, dy)} \tag{17}$$

where $l(y)$ represents the light intensity at point y and $M'(x, dy)$ denotes the free motion of the robots, assumed Gaussian with zero mean and covariance matrix $\Sigma_2 = [4, 0; 0, 4]$, expressed in square meters. We assume that, at each time step, a robot has a survival probability of $\epsilon = 1 - 10^{-3}$. The first equation of the specialised S/P model of (7) is written as follow:

$$\gamma_{t+1} = \gamma_t \epsilon\, Q + \mu_{t+1} \tag{18}$$

Furthermore, we assume that robots enter in the region from time $t = 0\,\mathrm{s}$ to $t = 25\,\mathrm{s}$ and from time $t = 100\,\mathrm{s}$ to $t = 125\,\mathrm{s}$. The number of new robots entering at each time step during these intervals has a Poisson distribution with mean 2. The decoupling of the intensity γ_t into the pair $(\gamma_t(1), \eta_t) \in (\mathbb{R}_+ \times \mathcal{P}(E))$ corresponding to the mass $\gamma_t(1)$ and to density function $\eta_t(dx_t) = \gamma_t(dx_t)/\gamma_t(1)$ as described in section VI-A can be written as:

$$\begin{cases} \gamma_{t+1}(1) & = \gamma_t(1)\epsilon + \mu_{t+1}(1) \\ \eta_{t+1} & = \beta_t \Psi_\epsilon(\eta_t)Q + (1-\beta_t)\bar{\mu}_{t+1} \end{cases} \tag{19}$$

where the first equation is simply the integral of Eq. (18) over the state space and the second is obtained dividing Eq. (18) by $\gamma_{t+1}(1)$ and using the definition of the operator $\Psi_g(\eta_t)$ given in Eq. (1):

$$\begin{aligned} \eta_{t+1} &= \frac{\gamma_{t+1}}{\gamma_{t+1}(1)} = \frac{\gamma_t \epsilon Q + \mu_{t+1}}{\gamma_{t+1}(1)} \\ &= \frac{\gamma_t(1)\eta_t \epsilon Q + \mu_{t+1}}{\gamma_{t+1}(1)} = \frac{\gamma_t(1)\eta_t \epsilon \frac{\eta_t(\epsilon)}{\eta_t(\epsilon)} Q + \mu_{t+1}}{\gamma_{t+1}(1)} \\ &= \frac{\gamma_t(1)\epsilon}{\gamma_{t+1}(1)} \Psi_\epsilon(\eta_t)Q + \frac{\mu_{t+1}(1)\bar{\mu}_{t+1}}{\gamma_{t+1}(1)} \end{aligned} \tag{20}$$

By using the notation introduced in Sec. III and letting $\bar{\mu}_{t+1} = \mu_{t+1}/\mu_{t+1}(1)$ and $\beta_t = \gamma_t(1)\epsilon/\gamma_{t+1}(1)$, the second equation of the system given in Eq. (19) is obtained.
The approximation of the solution of (19) is illustrated in Figure 6. At each step, with probability β_t, particles are resampled according to $\Psi_\epsilon(\eta_t)$ and are propagated according to Q. With probability $1 - \beta_t$, particles are resampled from $\bar{\mu}_{t+1}$. Algorithm 1 provides the details on how the numerical solution is obtained.

B. A model of swarm aggregation

In this section, the S/P model is specialized to describe the phenomenon of aggregation in robots moving in a square region of 100×100 m^2. In this region there are two square green areas that the robots can perceive when they are on them—see Fig. 7. The robots are required to aggregate in these areas. The initial distribution of the swarm is assumed to be known, its intensity Gaussian with mean $\gamma_0^\mu = [50, 45]^T$ m and covariance matrix $\gamma_0^\Sigma = 20\ \mathbf{I}_2\,\mathrm{m}^2$. The system is as follows:

$$\begin{cases} \gamma_{t+1} & = \gamma_t Q^{\pi_t} \\ \pi_{t+1} & = C_{\gamma_t} \end{cases} \tag{21}$$

where Q^{π_t} is defined in Eq. (4). The modification of the potential occurs when robots sense the green areas. These areas stimulate the robots to emit light signals that can be perceived by other robots. In this context, the potential is the intensity of the light.
In this case study, terms modeling dissipation or diffusion are thus not required. Moreover, the contribution term C_{γ_t} does not depend on the intensity of the potential π_t, but only on the expected intensity of robots in the aggregation area. Robots move according to Eq. (4) with parameters $\lambda = 10^5$, $Q_0(x, \cdot) = \mathcal{N}(\cdot; x, \Sigma_0)$, $Q_1(x, \cdot) = \mathcal{N}(\cdot; x, \Sigma_1)$ where $\Sigma_0 = 5\ \mathbf{I}_2$ m^2 and $\Sigma_1 = \mathbf{I}_2$ m^2. Each robot moves randomly but when a light stimulus is perceived, random motions towards regions of the space where the intensity of

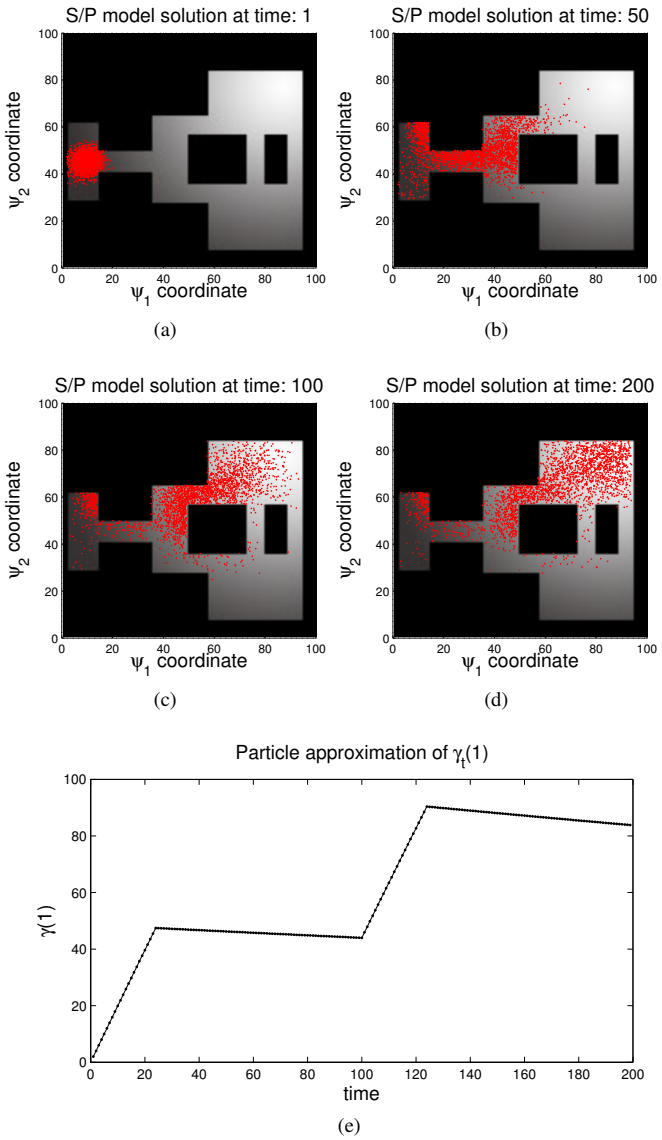


Fig. 6: Normalized intensity function η_t of the point process associated to the robot swarm at time $t = 1$ (a), $t = 50$ (b), $t = 100$ (c), and $t = 200$ (d), approximated by 5000 particles. The average number of active robots at each time step is given by $\gamma_t(1)$, shown in (e).

the light is greater have a higher probability. The probability of moving towards a light source is proportional to the intensity of the light itself. The release of the potential is modeled by the term π_{t+1} that can be approximated by:

$$\pi_{t+1}(dy) \approx \frac{\gamma_t(1)}{N} \sum_{i=1}^{N} \delta_{x_i^t}(A) C(x_i^t, dy) \qquad (22)$$

with $A \subset \mathbb{R}^2$ modeling the aggregation areas and $C(x_i^t, dy)$ capturing the intensity of the light perceived in the infinitesimal region dy emitted by a robot with state x_i^t. These terms are modeled as Gaussians, centered in x_i^t, whose covariance

Algorithm 1 Particle approximation algorithm

N: number of particles
$\xi_{i,t}^N$: i-th particle approximating γ_t, $i \in \{1, \dots, N\}$
η_t^N: particle approximation of η_t: $\eta_t^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{\xi_{i,t}^N}$
μ_t^N: particle approximation of μ_t
$g(x)$: function modeling the probability of a robot survival at x. For scenario VII-A $g(x) = \epsilon$
M: max number of iterations
Initialization
$\gamma_0^N(1) = \mu_0(1)$
for $i = 1$ **to** N **do**
  Sample $\xi_{i,0}^N$, $i \in \{1, \dots, N\}$ from $\bar{\mu}_0 = \mu_0(\cdot)/\mu_0(1)$
end for
Main loop
while $t \leq M$ **do**
  $\eta_t^N(g) = (1/N) \sum_{i=1}^{N} g(\xi_{i,t}^N)$;
  $\beta_t = (\gamma_{t-1}^N(1)\eta_t^N(g))/(\gamma_{t-1}^N(1)\eta_t^N(g) + \mu_t^N(1))$
  Compute the weight of each particle:
  $w_{i,t}^N = g(\xi_{i,t}^N)/\sum_i g(\xi_{i,t}^N)$
  for $i = 1$ **to** N **do**
    Sample $u \sim U[0, 1]$
    if $u \leq \beta_t$ **then**
      Sample a new particle from the current particle approximation with probability given by the weights $\bar{\xi}_{i,t}^N \sim w_{j,t}^N$, $j \in \{1, \dots, N\}$ and propagate it according to the dynamical model: $\xi_{i,t}^N \sim Q(\bar{\xi}_{i,t}^N, .)$
    else
      Sample a new particle from the birth distribution $\xi_{i,t}^N \sim \bar{\mu}_t$
    end if
  end for
  $\gamma_t^N(1) = \gamma_{t-1}^N(1)\eta_t^N(g) + \mu_t^N(1)$
  $t = t + 1$
end while

matrices model the intensity of the light over the space. Figure 7 illustrates the numerical solution of the system when the swarm distribution is approximated with $N = 500$ particles.

C. Swarming behavior of Brownian particles towards preferential directions

While existing robotic technologies are mostly based on mechatronic hardware principles (i.e., robots are mechanical machines controlled by electronics), a novel approach consists in designing and fabricating robots based on soft matter that function on the basis of chemical principles [9]. Taking inspiration from single cell organisms like protozoa and bacteria, a chemical robot can be defined as an internally structured vesicle in the size range of 10-100 μm, with the ability to uptake, store, chemically process and release molecules as a function of external stimuli. Prototypes of such chemical robots are already being built [30], [42], [43], [44]. In recent works on chemical robots such as [9], chemical robots are modeled as particles capable of releasing chemical signals that can be sensed by other particles in close proximity. In

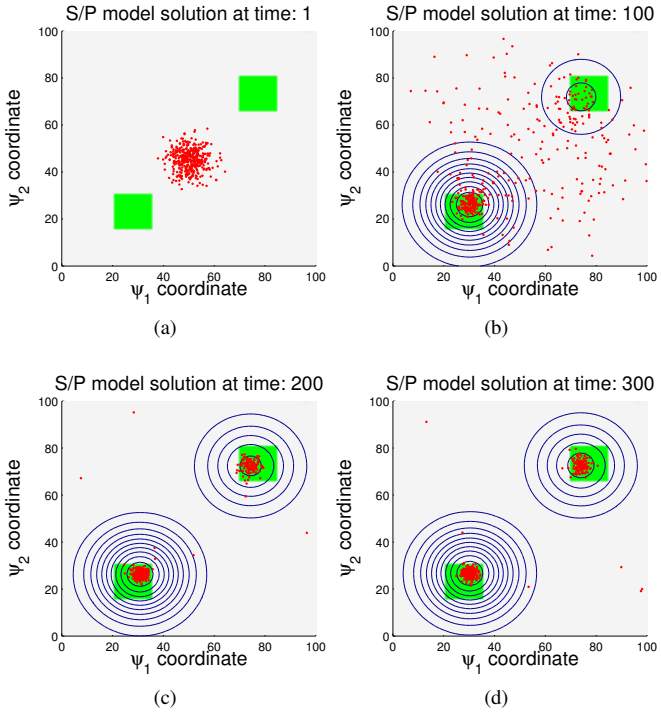


Fig. 7: Solution of the S/P model specialized as in Eq. (21) at time $t = 1$ (a), $t = 100$ (b), $t = 200$ (c), and $t = 300$ (d), approximated by 500 particles. Particles represent the discrete approximation of the intensity functions. Isolines of the Gaussian mixture corresponding to the potential are shown in blue.

this section we design the transport operator and the potential function that generate a phenomenon of coordinated motion towards a preferential direction in a swarm of chemical robots. The following version of the specialised S/P model of Eq. (8) will be studied numerically:

$$\begin{cases} \gamma_{t+1} & = \gamma_t Q^{\pi_t} \\ \pi_{t+1} & = \alpha \pi_t D + C_{\pi_t,\gamma_t} \end{cases} \tag{23}$$

In Eq. (23) it is assumed that no new particles enter the scene ($\mu_{t+1} = 0$) and that Q^{π_t} has the expression as in Eq. (4). The particles can move anywhere on the plane but are observed in a region of 250×300 mm^2. The random dynamics of the individuals are modeled trough Gaussian operators: $Q_0(x,\cdot) = \mathcal{N}(\cdot; x, \Sigma_0)$, $Q_1(x,\cdot) = \mathcal{N}(\cdot; x, \Sigma_1)$ and

$$\Sigma_0 \;=\; \sigma_0^2 \mathbf{I}_2 \text{ mm}^2 \quad , \quad \Sigma_1 = \sigma_1^2 \mathbf{I}_2 \text{ mm}^2$$

where $\mathbf{I}_2$ denotes the identity matrix of dimension two, $\sigma_0 = 9$ mm and $\sigma_1 = 6$ mm. The sensitivity threshold is set to $\lambda = 10^6$, and the diffusion of the potential occurs at a rate of $\alpha = 0.8$. The diffusion dynamics is Gaussian with mean zero and covariance $\sigma_D^2 \mathbf{I}_2$ with $\sigma_D = 6.3$ mm. According to Eq. (4), the robot dynamics is modeled as follows: if random movements chosen according to $Q_0(x,\cdot)$ are directed towards regions of the space associated to high potentials, the movements are performed, otherwise a random displacement is chosen according to $Q_1(x,\cdot)$. Deplacements towards regions with lower values of the potential are done with an exponentially decreasing probability.

The colloidal particles are able to modify the level of the potential in the environment by releasing chemical signals. The release is assumed to occur in a preferential direction. Although the implementation of this behavior on chemical robots has not been achieved yet, efforts to devise the technology necessary to coordinate the collective motion of particles and to orientate them in a certain direction are being made [9]. In the model, we assume the presence of an external force field which can orient the particles in such a way that the release of the chemical substances occurs in a specific direction. The response of the robots to the potential (second term of Eq. (23)) is approximated as follows:

$$C_{\pi_t,\gamma_t}(dy) \approx \frac{1}{N} \sum_{i=1}^{N} w_{x_i^t} \bar{C}_{\pi_t}(x_i^t, dy) \tag{24}$$

with $\bar{C}_{\pi_t}(x_i^t,\cdot) = \mathcal{N}(\cdot; x_i^t + c_0, \sigma_c^2 \mathbf{I}_2)$, having $c_0 = [0,5]^T$ mm and $\sigma_c^2 = 4.5$ mm^2. The release is done only if a particle perceives a sufficiently high concentration of chemical signal in its surroundings. The weight $w_{x_i^t}$ is written as:

$$w_{x_i^t} = \begin{cases} 0 & \text{if} \quad \pi_t(x_i^t) < \theta \\ \gamma_t(1) C_{\pi_t}(1) & \text{if} \quad \pi_t(x_i^t) \geq \theta \end{cases} \tag{25}$$

where the amount of potential injected by a particle is $C_{\pi_t}(1) = 1$. The sensing threshold is set to $\theta = 2 \cdot 10^{-4}$. The numerical solution of Eq. (23) is shown in Fig. 8. The solution is approximated with 500 particles. The initial potential is modeled as a Gaussian centered in $\pi_0^\mu = [150, 150]^T$ mm with covariance matrix $\pi_0^P = [120, 0; 0, 120]$ mm^2. The initial distribution of the swarm is modeled with a Gaussian term with mean $\gamma_0^\mu = [180, 175]^T$ m and covariance matrix $\gamma_0^\Sigma = 1000\ \mathbf{I}_2$ m^2. Figure 9 illustrates the phase transition occurring with increasing values of the dissipation coefficient α. Highly dissipative chemical signals tend to disappear rapidly since the chemical robots have less opportunities to sense them and to release additional potential. When this happens, the initial potential dissipates completely and the coupling of the system disappears. By increasing the dissipation coefficient towards 1, the initial chemical signal is able to remain in the environment long enough to be perceived by the swarm. In particular, for the the parameters discussed above, values of $\alpha > 0.6$ trigger the flocking behavior: as a response to the stimulus, particles release additional chemicals in the environment and progressively start to aggregate and move cohesively in the direction of the potential release. In the simulation, the preferential direction corresponds to the ψ_2-axis.

VIII. Conclusion

In the paper, we introduced a measure-valued recursion for spatial swarm robotics models arising from an random finite set (RFS) interpretation of the state of a swarm at any given time. The premise of the application of the RFS framework to swarm robotics is that the collection of robots' states can

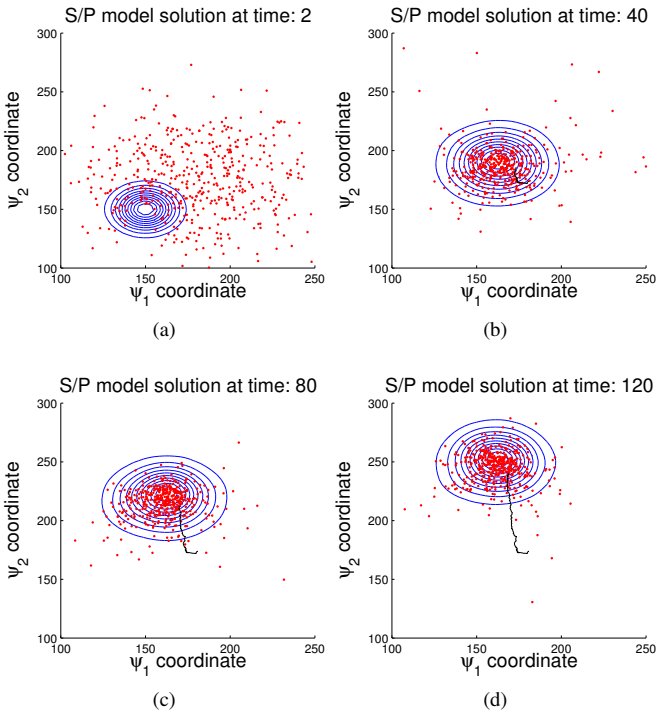


Fig. 8: Solution of the S/P model specialized as in Eq. (23) $t = 2$ (a), $t = 40$ (b), $t = 80$ (c), and $t = 120$ (d), approximated by 500 particles. Particles represent the discrete approximation of the intensity functions. Isolines of the potential are shown in blue.

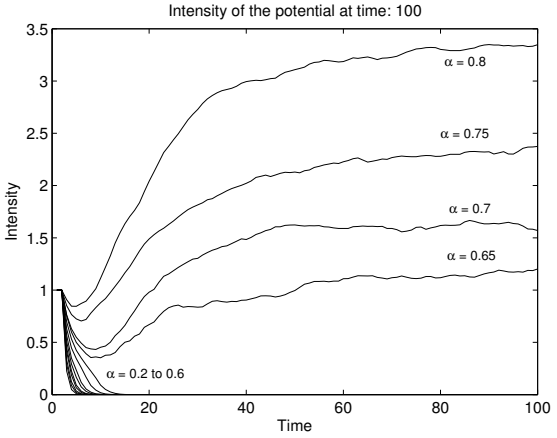


Fig. 9: Phase transition with varying dissipation coefficients α. Lower dissipation coefficients cause the potential to rapidly disappear. When the initial potential dissipates slowly, robots are able to sense it and can release additional potential and the process self-sustains. The intensity of the potential increases in time up to a value corresponding to a stable configuration of the system.

be treated as a random variable that is random both in the number and the value of the elements. When this is the case, the evolution in time of the intensity of the RFS can be used to describe the dynamics of robotic swarms in a computationally efficient way. We coupled the RFS-based modeling of the macroscopic behavior of a swarm of robots with a second general equation modeling the evolution in time of positive potentials. The resulting S/P model captures the dynamics of the interaction between the robots and the potential. Suitable forms for the operators of the abstract model have been derived for a series of scenarios. Results show that numerical solutions of the system are able to provide insights into the non-linear dynamics of complex phenomena such as coordinate motion and swarm aggregation. Moreover, the model allows for the study of the dynamics of swarm robotics systems, including phase transitions, when fundamental parameters are changed.

In the paper, numerical solutions have been derived by assuming that the potential function can be approximated at each time step by a weighted Gaussian mixture. While this assumption is introduced to render the models computationally tractable, it is also realistic in practical terms. Typical sensing operations encountered in swarm robotics, such as the recording of images or the perception of physical objects through infrared sensors, are often represented in Gaussians terms as a practical approximation or due to the noise in the acquisition process. More advanced robot swarms, such as chemical robots, will be able to sense chemical substances that exhibit diffusion dynamics whose nature is typically Gaussian. In more complex scenarios in which this assumption does not hold, a particle approximation of the evolution of the potential is required.

REFERENCES

[1] E. J. H. Robinson, N. R. Franks, S. Ellis, S. Okuda, and J. A. R. Marshall, "A simple threshold rule is sufficient to explain sophisticated collective decision-making," *PLoS ONE*, vol. 6, no. 5, p. e19981, 2011.

[2] T. D. Seeley, S. Camazine, and J. Sneyd, "Collective decision-making in honey bees: how colonies choose among nectar sources," *Behavioral Ecology and Sociobiology*, vol. 28, no. 4, pp. 277–290, 1991.

[3] G. Matheron, *Random Sets and Integral Geometry*. J. Wiley, New York, 1975.

[4] I. R. Goodman, R. Mahler, and H. T. Nguyen, *Mathematics of Data Fusion*. Norwell, MA, USA: Kluwer Academic Publishers, 1997.

[5] R. Mahler, *Statistical Multisource-Multitarget Information Fusion*. Norwood, MA, USA: Artech House, Inc., 2007.

[6] H. Hamann and H. Wörn, "A framework of space-time continuous models for algorithm design in swarm robotics," *Swarm Intelligence*, vol. 2, no. 2-4, pp. 209–239, 2008.

[7] E. L. Keshet, "Mathematical models of swarming and social aggregation," in *Proceedings of the 2001 International Symposium on Nonlinear Theory and its Applications*, Miyagi, Japan, 2001, pp. 1–7.

[8] O. Soysal and E. Şahin, "A macroscopic model for self-organized aggregation in swarm robotic systems," in *Swarm Robotics*, ser. Lecture Notes in Computer Science, E. Şahin, W. Spears, and A. F. Winfield, Eds. Springer, Berlin, Germany, 2007, vol. 4433, pp. 27–42.

[9] P. Grančič and F. Štěpánek, "Swarming behavior of gradient-responsive brownian particles in a porous medium," *Physical Review E*, vol. 86, p. 011916, 2012.

[10] K. M. Manesh, S. Campuzano, W. Gao, M. J. Lobo-Castanon, I. Shitanda, K. Kiantaj, and J. Wang, "Nanomotor-based biocatalytic patterning of helical metal microstructures," *Nanoscale*, vol. 5, pp. 1310–1314, 2013.

[11] T. M. Allen and P. R. Cullis, "Drug delivery systems: Entering the mainstream," *Science*, vol. 303, no. 5665, pp. 1818–1822, 2004.

[12] K. Lerman and A. Galstyan, "A general methodology for mathematical analysis of multi-agent systems," USC Information Sciences, Marina del Rey, CA, USA, Tech. Rep., 2001.

[13] M. Montes de Oca, E. Ferrante, A. Scheidler, C. Pinciroli, M. Birattari, and M. Dorigo, "Majority-rule opinion dynamics with differential latency: A mechanism for self-organized collective decision-making," *Swarm Intelligence*, vol. 5, no. 3-4, pp. 305–327, 2011.

[14] M. Massink, M. Brambilla, D. Latella, M. Dorigo, and M. Birattari, "On the use of bio-pepa for modelling and analysing collective behaviours in swarm robotics." *Swarm Intelligence*, 2013, in press.

[15] W. Liu and A. F. T. Winfield, "Modeling and optimization of adaptive foraging in swarm robotic systems," *International Journal of Robotics Research*, vol. 29, no. 14, pp. 1743–1760, 2010.

[16] S. M. Berman, "Abstractions, analysis techniques, and synthesis of scalable control strategies for robot swarms," Ph.D. dissertation, University of Pennsylvania, Philadelphia, USA, 2010.

[17] H. Hamann, "Towards swarm calculus: universal properties of swarm performance and collective decisions," in *Swarm Intelligence. Proceedings of the 8th international conference ANTS 2012*, ser. Lecture Notes in Computer Science. Springer, Berlin, Germany, 2012, vol. 7461, pp. 168–179.

[18] D. Kendall, "Foundations of a theory of random sets," in *Stochastic Geometry*. J. Wiley, New York, 1963, pp. 322–376.

[19] J. Serra, "The Boolean model and random sets," *Computer Graphics and Image Processing*, vol. 12, pp. 99–126, 1980.

[20] F. Caron, P. Del Moral, A. Doucet, and M. Pace, "Particle approximation of the intensity measures of a spatial branching point process arising in multi-target tracking," *SIAM Journal of Control and Optimization*, vol. 49, no. 4, pp. 1766–1792, 2011.

[21] B.-T. Vo, "Random finite sets in multi object filtering," Ph.D. dissertation, The University of Western Australia, Australia, 2008.

[22] B.-N. Vo, S. S. Singh, and A. Doucet, "Sequential Monte Carlo methods for multi-target filtering with random finite sets," *IEEE Transactions On Aerospace and Electronic Systems*, vol. 41, no. 4, pp. 1224–1245, 2005.

[23] J. Mullane, B.-N. Vo, M. Adams, and B.-T. Vo, *Random Finite Sets for Robot Mapping and SLAM - New Concepts in Autonomous Robotic Map Representations*, ser. Springer Tracts in Advanced Robotics. Springer, 2011.

[24] D. Daley and D. Vere-Jones, *An Introduction to the Theory of Point Process*, 2nd ed. Springer, Berlin, Germany, 2003.

[25] P. Del Moral, G. W. Peters, and C. Vergé, "An introduction to particle integration methods: with applications to risk and insurance," *ArXiv e-prints*, 2012.

[26] P. Del Moral, "Nonlinear filtering using random particles," *Theory of Probability and Its Applications*, vol. 40, no. 4, pp. 690–701, 1995.

[27] ——, "Measure valued processes and interacting particle systems. application to non linear filtering problems," *Annals of Applied Probability*, vol. 8, pp. 438–495, 1996.

[28] F. Schweitzer, *Brownian Agents and Active Particles: Collective Dynamics in the Natural and Social Sciences*. Springer Series in Synergetics, 2007.

[29] W. Agassounon and A. Martinoli, "A macroscopic model of an aggregation experiment using embodied agents in groups of time-varying sizes," in *2002 IEEE International Conference on Systems, Man and Cybernetics*, 2002, vol. 2, pp. 250–255.

[30] J. Dohnal and F. Štěpánek, "Fabrication of composite microcapsules by drop-on-demand inkjet: effect of precursor composition on the process limits," *Chemical Engineering Science*, vol. 66, pp. 3829–3835, 2011.

[31] P. Del Moral, *Feynman-Kac formulae. Genealogical and Interacting Particle Systems with Applications*. Springer, New York, 2004.

[32] A. Budhiraja, P. Del Moral, and S. Rubenthaler, "Discrete time markovian agents interacting through a potential," *ESAIM: Probability and Statistics*, 2013, in press.

[33] M. Brambilla, E. Ferrante, M. Birattari, and M. Dorigo, "Swarm robotics: A review from the swarm engineering perspective," *Swarm Intelligence*, vol. 7, no. 1, pp. 1–41, 2013.

[34] S. Garnier, J. Gautrais, and G. Theraulaz, "The biological principles of swarm intelligence," *Swarm Intelligence*, vol. 1, no. 1, pp. 3–31, 2007.

[35] M. Dorigo, E. Bonabeau, and G. Theraulaz, "Ant algorithms and stigmergy," *Future Generation Computer Systems*, vol. 16, no. 8, pp. 851–871, 2000.

[36] P.-P. Grassé, "La reconstruction du nid et les coordinations interindividuelles chez Bellicositermes natalensis et Cubitermes sp. la théorie de la stigmergie: Essai d'interprétation du comportement des termites constructeurs," *Insectes Sociaux*, vol. 6, no. 1, pp. 41–80, 1959.

[37] M. Pace, "Random Finite Set Modeling of Swarm Robotics Systems and Roboplasms: Fundamentals," IRIDIA, Université Libre de Bruxelles, Brussels, Belgium, Tech. Rep., 2013.

[38] P. Del Moral, L. Kallel, and J. Rowe, "Modeling genetic algorithms with interacting particle systems," in *Theoretical Aspects of Evolutionary Computing*. Springer, Berlin, Germany, 2001, pp. 10–67.

[39] Y.-C. Ho and R. Lee, "A Bayesian approach to problems in stochastic estimation and control," *IEEE Transactions on Automatic Control*, vol. 9, no. 4, pp. 333–339, 1964.

[40] B. Ristic, S. Arulampalam, and N. Gordon, *Beyond the Kalman Filter: Particle Filters for Tracking Applications*. Norwood, MA, USA: Artech House, 2004.

[41] J. D. Bjerknes, A. F. Winfield, C. Melhuish, and C. Lane, "An analysis of emergent taxis in a wireless connected swarm of mobile robots," in *In IEEE Swarm Intelligence Symposium*. IEEE Press, 2007, pp. 45–52.

[42] S. Balasubramanian, D. Kagan, C. Hu, S. Campuzano, M. Lobo-Castaon, N. Lim, D. Kang, M. Zimmerman, L. Zhang, and J. Wang, "Micromachine-enabled capture and isolation of cancer cells in complex media," *Angewandte Chemie International Edition*, vol. 50, no. 18, pp. 4161–4, 2011.

[43] J. Wu, S. Balasubramanian, D. Kagan, K. M. Manesh, S. Campuzano, and J. Wang, "Motion-based DNA detection using catalytic nanomotors," *Nature Communications*, vol. 1, no. 36, 2010.

[44] D. Kagan, R. Laocharoensuk, M. Zimmerman, C. Clawson, S. Balasubramanian, D. Kang, D. Bishop, S. Sattayasamitsathit, L. Zhang, and J. Wang, "Rapid delivery of drug carriers propelled and navigated by catalytic nanoshuttles," *Small*, vol. 6, no. 23, pp. 2741–2747, 2010.