A Purely Arithmetical, yet Empirically Falsifiable, Interpretation of Plotinus' Theory of Matter

Bruno Marchal, IRIDIA, Université Libre de Bruxelles, marchal@ulb.ac.be

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Résumé

The self-analysis abilities of formal theories or theorem provers are outstanding. We show that the study of self-observing "ideal" machine leads to natural arithmetical interpretations of the *hypostases* that Plotinus discovered by looking inward. Those corresponding to his "Matter Theory" are compared with the logic of empirical current physics.

Keyword : Gödel, mechanism, modal logic, provability, consistency, quantum logic, intuitionist logic, Plotinus, hypostases.

1 Incompleteness and Mechanism

There is a vast literature where Gödel's first and second incompleteness theorems are used to argue that human beings are different of, if not superior to, any machine. The most famous attempts have been given by J. Lucas in the early sixties and by R. Penrose in two famous books [53, 54]. Such type of argument are not well supported. See for example the recent book by T. Franzèn [21]. There is also a less well known tradition where Gödel's theorems is used in favor of the mechanist thesis. Emil Post, in a remarkable anticipation written about ten years before Gödel published his incompleteness theorems, already discovered both the main "Gödelian motivation" *against* mechanism, and the main pitfall of such argumentations [17, 55]. Post is the first discoverer[1] of *Church Thesis*, or Church Turing Thesis, and Post is the first one to prove the first incompleteness theorem from a statement equivalent to Church thesis, i.e. the existence of a universal—Post said "complete"—normal (production) system[2].

In his anticipation, Post concluded at first that the mathematician's mind or that the logical process is essentially creative. He adds :

> "It makes of the mathematician much more than a clever being who can do quickly what a *machine* could do ultimately. We see that a machine would never give a complete logic ; for once the machine is made we could prove a theorem it does not prove"(Post emphasis).

But Post quickly realized that a machine could do the same deduction for its own mental acts, and admits that :

> "The conclusion that man is not a machine is invalid. All we can say is that man cannot construct a machine which can do all the thinking he can. To illustrate this point we may note that a kind of machine-man could be constructed who would prove a similar theorem for his mental acts."

[1] A case can be made that Babbage could have discovered it too, through his invention of a functional notation capable of describing his analytical engine. According to Lafitte 1932 the old Babbage was more proud of his language than of his analytical engine [32].

[2] The informal but rigorous proof can be given in a footnote : let N be the set of natural numbers $\{0, 1, 2, 3, ...\}$. Let us say that a function from N to N is computable if we can describe in a finite way how to compute it, in a language with a checkable grammar. Church thesis asserts the existence of *a universal language*, that is, a language in which we can describe *all*, but not necessarily *only*, computable functions from N to N. Suppose now the existence of a *complete* theory T about Arithmetic, or, more easily, about machines or codes. We will get a contradiction. Given the checkability of the grammar, we can enumerate all the codes : $C_0, C_1, C_2, C_3, ...$ accepting one input, in the universal language. If T is a complete theory, T would be able to decide, for each i, if C_i is defined on each n or not. From this, by the use of the complete theory T, we can now enumerate the total (always defined) computable function $f_0, f_1, f_2, f_3, ...$ which, by Church thesis are all among the functions defined by the codes $C_0, C_1, C_2, C_3, ...$ But then the diagonal function g defined by $g(n) = f_n(n) + 1$ is computable. Thus, there is a number k such that $g = f_k$. But then $g(k) = f_k(k) = f_k(k) + 1$. Given that the f_k are total functions, $f_k(k)$ is a well defined number, and we can subtract it on both sides, so that $0 = 1$. So, either there is no universal language and thus no universal machine capable of understanding them—and Church thesis is false—or there is no complete theory for numbers or machines.

This has probably constituted his motivation for lifting the term *creative* to his set theoretical formulation of mechanical universality [56]. To be sure, an application of Kleene's second recursion theorem, see [30], can make any machine self-replicating, and Post should have said only that man cannot *both* construct a machine doing his thinking *and proving* that such machine do so. This is what remains from a reconstruction of Lucas-Penrose argument : if we are machine we cannot constructively specify which machine we are, nor, a fortiori, which computation support us. Such analysis begins perhaps with Benacerraf [4], (see [41] for more details). In his book on the subject, Judson Webb argues that Church Thesis is a main ingredient of the Mechanist Thesis. Then, he argues that, given that incompleteness is an easy—one double diagonalization step, see above—consequence of Church Thesis, Gödel's 1931 theorem, which proves incompleteness *without* appeal to Church Thesis, can be taken as a confirmation of it. Judson Webb concludes that Gödel's incompleteness theorem is a very lucky event for the mechanist philosopher [70, 71]. Torkel Franzèn, who concentrates mainly on the negative (antimechanist in general) abuses of Gödel's theorems, notes, after describing some impressive self-analysis of a formal system like Peano Arithmetic (PA) that :

> "Inspired by this impressive ability of PA to understand itself, we conclude, in the spirit of the metaphorical "applications" of the incompleteness theorem, that if the human mind has anything like the powers of profound self-analysis of PA or ZF, we can expect to be able to understand ourselves perfectly".

Now, there is nothing metaphorical in this conclusion if we make clear some assumption of classical (platonist) mechanism, for example under the (necessarily non constructive) assumption that there is a substitution level where we are turing-emulable. We would not *personally* notice any digital functional substitution made at that level or below [38, 39, 41]. The second incompleteness theorem can then be conceived as an "exact law of psychology" : no consistent machine can prove its own consistency from a description of herself made at some (relatively) correct substitution level—which exists by assumption (see also [50]). What is remarkable of course is that all machine having enough provability abilities, can prove such psychological laws, and as T. Franzèn singles out, there is a case for being rather impressed by the profound self-analysis of machines like PA and ZF or any of their consistent recursively enumerable extensions[3]. This leads us to the positive—open minded toward the mechanist hypothesis—use of incompleteness. Actually, the whole of recursion theory, mainly intensional recursion theory [59], can be seen in that way, and this is still more evident when we look at the numerous application of recursion theory in theoretical artificial intelligence or in computational learning theory. I refer the reader to the introductory paper by Case and Smith, or to the book by Osherson and Martin [14] [46]. In this short paper we will have to consider machines having both provability abilities *and* inference inductive abilities, but actually we will need only trivial such inference inductive abilities. I call such machine "Löbian" for the proheminant rôle of Löb's theorem, or formula, in our setting, see below.

Now, probably due to the abundant abuses of Gödel's theorems in philosophy, physics and theology, negative feelings about *any* possible applications of incompleteness in those fields could have developed. Here, on the contrary, it is our purpose to illustrate that the incompleteness theorems and some of their generalisations, provide a rather natural purely arithmetical interpretation of Plotinus' Platonist, non Aristotelian, "theology" including his "Matter Theory".

As a theory bearing on matter, such a theory is obviously empirically falsifiable : it is enough to compare empirical physics with the arithmetical interpretation of Plotinus' theory of Matter. A divergence here would not refute Plotinus, of course, but only the present arithmetical interpretation.

This will illustrate the internal consistency and the external falsifiability of some theology. Here the term "theology" can be interpreted in some general, albeit non necessarily physicalist, sense of "theory of everything", or "truth theory", including what subjects can prove, or known, or guess about themselves and their possible neighborhoods, and what is true about them but which they cannot prove, but still guess, or not. This could hopefully help to eventually unify fundamental fields like some axiomatic theologies, theoretical physics, theoretical computer science and number theory[4]. By incompleteness machine's (pure) theology could already have been defined by *true* computer science *minus* computer's computer science. This will be made more precise below by the use of Solovay theorem [61].

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[3] I identify Peano Arithmetic, Zermelo Fraenkel and other axiomatisable theories with their theorem provers. A theorem by Craig can justify this move, see Boolos and Jeffrey [7]. Thus I will say that p is proved *by* PA, instead of saying the usual "p is proved *in* PA.

[4] The number theoretical aspect of computer science is beyond the scope of the present paper, but the basic bridge has been provided by the works of Davis, Robinson and Putnam and Matiyasevich which singles out the existence of universal diophantine polynomials, see [47]. Note that Diophantus is contemporary of Plotinus. Both have been taught by Hypatia one century later in Alexandria [18].

2 Plotinus and Machine's Methodologies

Plotinus makes clear that its methodology belongs to those among the platonists who are both rationalist and mystic (looking inward). His basic methodology consists in self-analysis together with rational analysis and communication in a way inspired mainly by Plato and Aristotle. Our own methodology will consist in studying what "platonist" universal machine, the definition is given below, having enough deducibility and inferability abilities, can discover by self-analysis.

It is hard to be fair with Plotinus, third century A.D., by summing up his work in a paragraph[5]. At the same time Plotinus is enough clear and so much different from the current widespread Aristotelian conception of reality, that this project will not suffer too much from the obvious simplification it is asked for, and for which I apologize in advance. It is hoped that the self-observing machine's discourses will appear to be near neoplatonists like Plotinus or Proclus [63, 69].

Plotinus' view of "reality" can be given in term of three main hypostases. According to MacKenna, those hypostases are generalized and abstract notions of *Persons* [34]. Each hypostasis can be considered as a view of "reality" or "truth" from a personal, although "divine" (i.e. true but non effective, see below), points of view.

The three main primary hypostases are : the One, the Divine Intellect or Intelligible Realm (Plato's Nous), and the All-Soul or Universal-Soul. And there are also what I will call, to be short, the two *secondary "hypostases"* : the Intelligible Matter and the Sensible Matter. If we take into account the differences between the discursive, terrestrial, discourses and the true or divine possible discourses, this makes a total of a priori 5 times 2 = 10 "hypostases", in a more general sense than Plotinus' use of that word[6]. Note that Plotinus considers that the One and the Matter, associated to the secondary "hypostases", are above, respectively below, the realm of "existing things" or "authentically existing things", which concerns mainly the divine intellect's ideas (Plato's *Nous*).

The One can be considered as the ineffable, non necessarily effective, transcendental origin or source of everything. The word "origin" is closer to a mathematical or arithmetical origin than to a spatiotemporal cause[7]. As a *person*, the One can be considered as sustaining a degenerate zero person point of view, comparable to Nagel's point of view from nowhere [51]. The One is thus the ultimate fundamental "reality" responsible of the existence of anything capable of existence. It contains implicitly the other primary hypostases, sometimes described as different phases of the One. The One is also called "Good" in the sense that it will serve as a sort of universal attractor of the (terrestrial) souls. This implies a kind of two-way cosmogony. The One produces originally the divine intelligible which produces the universal and discursive souls which produce, by contemplation, Nature and eventually Matter—most of the time identified with "Evil" by the Platonist ; but, from the inside personal views, it looks the other way round. Somehow the terrestrial souls feel as if they were extracting themselves from Evil-Matter to tend toward the One. This is probably why Plotinus was an optimistic philosopher—Porphyry called him once the happy Plotinus.

The second hypostase, the "Intelligible" or "Divine Intellect" is mainly Plato's Nous, i.e. his world of intellectual or immaterial ideas. It is related with the *logos*, as either terrestrial or divine verbs or discourses. What is considered as "existing", or "authentically existing" are the ideas belonging to the divine intellect. Plotinus is priviledging some passage of Plato's Parmenides, by making the intelligible realm second to the One, and argues, against Aristotle, that the ultimate One cannot be a thinking subject. Plotinus argues indeed that thinking already needs some more primitive reality, by having to divide it into thinking subject and object of thinking.

The third hypostase, the "All-Soul" hypostasis, appears as a way of combining both the One and the Intellect, and thus allowing them to participate in one principle. Somehow the All-Soul is a version of an intellect which keeps better its ground through a direct link with the innefable One, and as such inherits part of its ineffability. The All-Soul is responsible for the existence of (subjective) time and of the creation of Nature and eventually Matter through a process of contemplation. This is the opposite of the Aristotelian metaphysics where somehow, mind and person arise or emerge from the organisation of some primitive or primary matter[8]. As a rationalist, Plotinus does not hide some difficulties entailed by his approach, notably concerning the rôle of the soul and its relative place with respect to the secondary "hypostases".

[5]We have used the new, unabridged, translation of Plotinus' Enneads by MacKenna (Larson Publications, 1992), together with the classical translation of A.H. Amstrong 1966 (Classical Loeb Library, Harvard University Press) and the older french translation by Emile Bréhier (Collection *Les Belles Lettres*, Paris, 1924q).

[6]Note that Plotinus strictly reserves the term "hypostasis" for the three primary, and divine, hypostases.

[7]See [72] for a physicist argument that the origin of the physical laws can hardly be physical.

[8]See [41], see also [10].

The material or secondary "hypostases" corresponds to the "Two Matters" of the second Ennead[9] (II, 4) : the intelligible matter and the sensible matter, most of the time described as the matter "there", meaning in the divine Nous, and the matter "here", meaning that it has a sensible component for the terrestrial, intellectual, or discursive souls. Matter itself is then, following Aristotle, described by indeterminateness and/or privation. Plotinus departs from Aristotle by defining literally that matter, exclusively and quasi-axiomatically, by this very notion of privation or indeterminateness. This makes matter prone to acquire or to represent distinctive and possibly alternate incidental (contingent) qualities, but in a such a way that matter itself remains invariant and separated from any of those qualities. This makes Matter literaly the opposite or the negation of the intelligible. Plotinus refers to Plato's Timaeus for the need here of a "bastard" or "spurious" reasoning to operate on that theoretical *Unintelligibility*.

I will illustrate that, thanks to their outstanding self-reference powers, the correct or honest Löbian machine cannot escape the discovery of an arithmetical version for each of those hypostases, mainly as intensional variants of provability[10]. Such intensional variants are made necessary by the incompleteness phenonomena. Such variants will include the "material" hypostases. They will be described in a precise way through their transparent arithmetical interpretations, and they will justify some precise and empirically testable logics of observability, where the "spurious reasoning" should lead to an arithmetical measure of probability or credibility. In fact each arithmetical hypostasis will give rise to weak logics[11] structuring differently, "from inside", or from "personal points of view", the arithmetical reality.

Now, physicists have been led in the last century to non boolean logics of observability, known as quantum logics [15, 16, 26, 49, 3]. Such logics capture many counter-intuitive propositions for which we were not prepared by the observation of the "macro-world" which provides an apparent canonical boolean phase spaces relating classical physics to classical logics. Like the whole of quantum physics, such logics are not easy to interpret, but our approach is mainly formal so that we will avoid any prematured interpretation problems. This is made possible by a result of R. Goldblatt (see [23]) showing that a minimal (propositional) quantum logic MQL can be translated in the classical modal logic, known as **B**.

Theorem 1 : MQL proves A iff **B** proves $t_{mql}(A)$.

B is generated by the closure of the set $\{K, \Box p \rightarrow p, p \rightarrow \Box\Diamond p\}$ for the modus ponens rule MP and the necessitation rule NEC (derive $\Box p$ from a derivation of p). K is for the "Kripke formula"

$\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$, and t_{mql} is a translation, called quantization [57], from quantum propositional quantum formula to classical modal formula : the quantization $t_{mql}(p)$ of an atomic formula p is given by the classical modal formula $\Box\Diamond p$, where "$\Diamond p$" is an abbreviation of $\neg\Box\neg p$. The quantization of $\neg A$ is given by the application of the box $\Box$ applied to the negation of the quantization of A, and quantization commutes with conjonction.

This result is similar to the presumption by Gödel [22] that the typical "introspective knower" modal logic S4, which follows from $\{K, \Box p \rightarrow p$ (incorrigibility), $\Box p \rightarrow \Box\Box p$ (introspection) $\}$, by application of the Modus ponens and necessitation rules, formalizes soundly and completely, in the classical frame, the Heyting-Brouwer (propositional) intuitionist logic INT. The result has been proved by McKinsey and Tarski[48] :

Theorem 2 : INT proves A iff S4 proves $t_{int}(A)$,

and has been made still stronger by Grzegorczyk[25] :

Theorem 3 : INT proves A iff S4Grz proves $t_{int}(A)$,

where S4grz is the system S4 with the addition of the Grz formula $\Box(\Box(p \rightarrow \Box p) \rightarrow p) \rightarrow p$.

$t_{int}(A)$ is Gödel 1933 translation [22] : $t_{int}(\neg p) = \neg\Box p, t_{int}(p \rightarrow q) = \Box p \rightarrow \Box q$,

$t_{int}(p \wedge q) = p \wedge q, t_{int}(p \vee q) = \Box p \vee \Box q$.

This is very appealing for the present approach, mainly due to a theorem by Solovay generalising Gödel's incompleteness theorem by two startling *completeness* theorems : the modal logics G and G*

[9] See [13, 68] for a larger treatment.

[10] A case can been made, through an analysis of his insightful treatise *on number* (Ennead VI,6) that Plotinus could have welcome such or similar enterprise.

[11] A propositional logic is weak when the set of its theorems is properly included in the set of the classical tautologies.

formalise completely the provable provability logic, and the true provability logic. Omitting usual substitution rules, here is a formal presentation of G, build on classical propositional calculus. L denotes the formula of Löb[12] [33]. Note the absence of the necessitation Rule NEC for G*. G is given by :

$$
\begin{array}{lll}
\text{AXIOMS} : & \Box(A \to B) \to (\Box A \to \Box B) & \text{K} \\
 & \Box A \to \Box\Box A & 4 \\
 & \Box(\Box A \to A) \to \Box A & \text{L} \\
\text{RULES} : & \frac{A\ ,\ A \to B}{B} & \text{MP} \\
 & \frac{A}{\Box A} & \text{NEC}
\end{array}
$$

and G* is given by :

$$
\begin{array}{lll}
\text{AXIOMS} : & \text{Any Theorem of G} & \\
 & \Box A \to A & \text{T} \\
\text{RULES} : & \frac{A\ ,\ A \to B}{B} & \text{MP}
\end{array}
$$

Let us define an arithmetical realisation R by a function which assigns to each propositional letter $p, q, r...$ an arithmetical sentence. An arithmetical interpretation i of a modal formula is given recursively by a realisation R, for the atomic letter, i.e. $i(p) = R(p)$, i commutes recursively with the boolean connectors, and, $i(\Box p) = Bew(\ulcorner p \urcorner)$, i.e. Gödel celebrate provability (beweisbar in German) arithmetical predicate. Solovay first completeness theorem asserts that G proves A if and only if PA proves $i(A)$ for any arithmetical interpretation, i.e. for any arithmetical realisation R of the atomic letter. The second completeness theorem is that G* proves A iff $i(A)$ is true, i.e. true, for any realisation R, in the usual number theoretical sense, or true in the so-called standard model of PA.

G captures the provable sentence by the machine ; and G* captures the true one, including the non provable one. The propositions belonging to the set difference G* $\setminus$ G are still inferable by the machine. Indeed, G* can be shown to be decidable. Solovay proved that G* proves a formula F if and only if G proves that the conjunction of the "reflection formula" $\Box g \to g$ implies F, where g is any boxed subformula of F (boxed means having the shape $\Box x$), see [61]. Some of those provable/truth splittings are inheritated by some of the intensional variant of provability. Indeed, with $\Box p$ interpreted as Bew($\ulcorner p \urcorner$) with p an arithmetical proposition, although $\Box p \wedge p$, $\Box p \wedge \Diamond p$, $\Box p \wedge \Diamond p \wedge p$, are truly equivalent (as G* can prove), none are always, for any arithmetical p, provably so by the machine. This makes them obeying different modal logics having different weak logic interpretations.

Plotinus' hypostases will be (re)defined arithmetically through the use of those intensional nuances. This can be translated in the language of some universal machine similar to Peano arithmetic, and Goldblatt's theorem will give us a way to measure the degree of plausibility of this arithmetical version of Plotinus, by comparing the arithmetical interpretation of Plotinus matter hypostases with quantum logic.

Note that we do not allow the presence of free variable in the scope of the provability predicate. We contend ourself here with the "propositional" provability logic for which those completeness theorems hold. See [6] for proofs that the quantified version of G and G* are as undecidable as it can possibly be.

3 Weak Computationalism

Our strategy consists in interviewing a platonist "sufficiently introspective" chatty universal machine. By saying that the machine is "platonist", we mean that the machine asserts (or proves, believes, etc.) the principle of excluded middle, among the classical tautologies. The letters $p, q, ...$ will always represent *arithmetical* propositions. Those correspond to the first order logical formula together with the usual symbols of formal arithmetic. By saying that the machine is universal, we mean that the machine is able to prove all true Σ_1-proposition p, i.e. arithmetical propositions which are provably equivalent (by the machine) to a proposition with the shape : $\exists x P(x)$ where P is a primitive recursive arithmetical predicate. Put in another way, it means that for any Σ_1-proposition p the proposition $p \to \Box p$ is true *for* the machine. It can be shown that such a machine has the full power of a Universal Turing Machine. By saying that the machine is "sufficiently introspective" we mean that, for any Σ_1-proposition p, not only $p \to \Box p$ is true *for* the machine, but is actually provable *by* the machine. Given that the Gödelian provability predicate, represented by the box $\Box$ is itself Σ_1, the machine is able to prove $\Box p \to \Box\Box p$ for

[12]See [44] for the importance of the Löb formula, which is a genuine generalisation of Gödel's second incompleteness, in a setting similar to the present paper.

any p. By "chatty" machine, we mean that the machine is so programmed that it dovetails on its proofs, and thus asserts soon or later all its provable propositions. Being classical and sufficiently rich, such a machine is Löbian, and its provability predicate, definable in its own language, is correctly formalized by the logics G and G*.

By "interviewing" some machine, we are implicitly assuming a very weak version of the computationalist hypothesis, or digital mechanism, in the cognitive science. But, a priori the machine itself is not supposed to assume the computationalist hypothesis. Still, during the interview itself, we will have to translate it explicitly in the language of the machine. Some thought experiments can justify that the available verifiable sort of reality for universal machine are determined by the true Σ_1 sentences[13]. So we get a computationalist version of G and G* by adding the axiom $p \to \Box p$ to the logic G and G*. This gives the corresponding logics V and V* which have been proved sound and complete for the (provable and true respectively) logic of provability and consistency of the Σ_1 sentences by A. Visser [67].

Our interview is an *infinite* conversation, made finite through the use of Solovay and Visser theorems. To say that G (resp V) proves $\Box p \to \Box\Box p$, means that the chatty machine asserts $\Box p \to \Box\Box p$ for any arithmetical (resp. Σ_1 arithmetical) interpretation of the formula p. It means for example that the machine tells us $\Box(\ulcorner 1+1=1 \urcorner) \to \Box(\ulcorner \Box(\ulcorner 1+1=1 \urcorner) \urcorner)$, but $\ulcorner 1+1=1 \urcorner$ can be substituted by any (false or true) arithmetical formula.

For reason of simplicity, we will confine to machine "talking arithmetics" ; but it is easy to generalized the result for richer machine like a theorem prover for ZF (Zermelo Fraenkel Set Theory), or even for non-mechanical entities like axiomatized version of second order logic with the (infinite) ω-rule. This follows from results easily accessible in the 1993 book by Boolos [6].

Actually we can associate, hopefully functorially[14], a Plotinian theology for each Löbian machine, but we could confine ourself to the "theology of a Peano Arithmetic machine", with PA seen as a generic typical simple Löbian machine. All the sound recursively enumerable extensions of PA admit the same formal modal theology. To be sure, G ang G* remains sound and even complete for much more general Löbian entity, see again [6] for more information.

4 The arithmetical interpretation of Plotinus' hypostases

Each hypostase will be interpreted by a set of arithmetical sentences.

Plotinus' *One* is interpreted by Arithmetical Truth, i.e the set of all true arithmetical sentences. In case we were interviewing ZF, we would have needed the more complex set-theoretical truth. In any case, it follows from Tarski theorem that such a truth set is not definable by the machine on which such truth bears. Nevertheless, she can already, but indirectly, point to its truth set by some sequence of approximations, and there is indeed a sense to say that Löbian machines are able to prove their own "Tarski theorem", illustrating again the self-analysis power of those theorem prover machines. See Smullyan's book [60] for a sketch of that proof and reference therein. In this sense we recover the "One" ineffability, and it is natural to consider arithmetical truth as the (non-physical) cause and ultimate reality of the arithmetical machine. This is even more appealing for a neoplatonist, than just a platonist, given the return of the neoplatonist to the Pythagorean roots of platonism [52]. The atomical verifiable "physical" proposition will be modelized by the Σ_1 sentences. Note that the machine *can* define the restricted, computationalist, notion of Σ_1-truth.

Plotinus' discursive *intellect* is interpreted by the machine Gödelian provability predicate *Bew* itself. The corresponding set of arithmetical sentences is the set of provable (by PA, or some Löbian machine M) arithmetical modal sentences. This set is captured by the modal logic G by the first half of Solovay theorem. By the second half of Solovay's theorem, there is a notable second set to consider : the set of the *true*[15] arithmetical interpretations of the modal formula, as single out by the modal logic G*. This provides a natural arithmetical interpretation of the "divine intellect". G play the rôle of discursive reason, or science about oneself as seen (or conjectured) as a finitely formal entity. G* plays the rôle of the whole (propositional) truth about the machine, including what is true but unprovable by the machine. This corresponds to a notion of true inference, as G* is decidable, and thus trivially "correctly

[13]Mainly the Universal Dovetailer Argument. The argument shows that the computationalist hypothesis entails self-duplicability, from which follows a notion of first person indeterminacy. It is shown that whatever means are used, if any, to quantify that indeterminacy, the quantification remains invariant for some transformation, making physical predictions relying on a (relative) measure on true Σ_1-propositions, [38, 39, 41, 42, 43, 44, 45].

[14]In that case we have to consider some interpretability logic [19]. Those logics constituted refinements of the provability logics. Algebraic approaches should help here [35].

[15]In the usual sense of elementary school. Equivalently p is true if p is satisfied by the "standard model of PA".

inferable", although it needs an act of faith, as the machine can prove to herself[16]. It is remarkable that, in this setting, reason (G) is included in faith (G*), so that only bad faith can fear reason. This is coherent with the scientific attitude of the pagan neoplatonist, and thus rationalist, theologians. The corona set $G^* \setminus G$ can represent the *pure* theology. This is a set, closed for the modus ponens rule, of the true but unbelievable, unprovable, unassertable by any self-referentially correct machine, propositions. Plotinus believed that the *divine intellect* has self-referential complete knowledge, but we cannot follow him here : only the discursive machine, capture by G has self-referential but incomplete knowledge. G* has complete knowledge, but not about itself, just about the machine it talks about. Actually, in the mechanist or arithmetical setting, Plotinus' critics about Aristotle attribution of self-thinking to the One can be repeated on the level of the divine intellect. This can perhaps be considered as a serious departure between the Löbian entities and Plotinus.

It is important to realize that both G and G* talk about the machine in a *third person way*. It corresponds to a situation where a computationalist practicionners is reasoning about himself after betting on some level of digital formal self-description, for example in term of a giant rational complex matrix (represented in the arithmetical language say) representing, hopefully, a reasonable approximation of its "brain" quantum state, whatever that brain is supposed to be.

Plotinus' *All-Soul*, at least in its discursive form, is captured by Plotinus, according to Bréhier, by the classical traditional and Theaetetical way for defining knowledge by true justified opinion : to know p is to believe (prove) p and p is true [11]. Such a stratagem is very much debated [12] and it can be related to both the mechanist hypothesis and the use of dream in metaphysics [36, 37, 2, 40, 41]. This arithmetical "All-Soul" cannot be represented directly in the (arithmetical) language of the machine. For example Tarski non definability of truth theorem forbids to define the knower by some expression like $\mathrm{Bew}(\ulcorner p \urcorner) \wedge \mathrm{True}(\ulcorner p \urcorner)$, given that the truth predicate "True" bearing on the machine cannot be defined by the machine. More generally, it can be shown that no predicate of knowledge (obeying for example the S4 modal axioms) can be defined formally in any self-referentially correct way [28], [62]. But we *can* define for each arithmetical p, the knowledge of p by the provability of p *and*, simply, p. In the formal mathematical setting this has been done independently by Boolos, Goldblatt, and Kuznetsov & Muravitsky in the USSR [31, 5, 24]. Eventually Artemov makes this "definition" a thesis and defends that it is comparable, as a thesis, to the Church thesis (which bears on computations), for the notion of informal proof [1]. It has indeed much in common with Brouwer "unformalizable" notion of creative subject [8, 9, 65, 41] which plays some rôle in the foundation of intuitionist mathematics.

Now, PA, in particular, is a sufficiently simple machine so that *we know* it is correct, and self-referentially correct when talking about itself. So, is it not obvious that $\Box p \rightarrow p$ is always (for all p) true ? Yes, it is, but PA can neither prove or know that. G* does indeed prove that $\Box p$ is equivalent with $\Box p \wedge p$, for any arithmetical (realisation) of p, but G does not prove it. For many arithmetical sentences, such equivalence belongs to the corona of true but unbelievable and thus unknowable, but still inferable, truth. It is also obvious that $(\Box p \wedge p) \rightarrow p$, so we see that $\Box p \wedge p$ defines an "incorrigible" first person notion of knowledge, and it has been proved that the logic S4Grz, see above, is sound and complete for both the provable and true point of view (see [6]). G and G* have exactly the same discourse *about* the first person, like if the knower, attached in this way to PA, was confusing truth and provability, in a manner similar to an intuitionist philosopher [6]. This can been made precise : by using the Gödel's translation (see above) we do find indeed an arithmetical interpretation of intuitionist logic. Semantical considerations about the Grz formula can be used to argue that this knower logic can be related to a subjective (antisymmetrical and possibly bifurcating and fusing) temporal logic, quite natural in view of the Plotinus's idea that the "Soul" generates time.

An intensional variant of the provability logic like S4Grz provides also a tool for reconstructing diverse uses of Gödel's theorem in the philosophy of mind. See [41] for an analysis of Lucas's "errors" and Benacerraf's reconstruction, through intensional variant of G and G*. Many confusions in this field can be recasted in term of confusion between *third person point of view* (treated by G and/or G*) and first person, singular or plural, points of view, treated through the S4Grz variants or those presented below.

[16] A "rich" Löbian machine, like ZF, can prove the whole (propositional) theology of a less rich Löbian machine, like PA, but has still to make an act of faith to lift that theology on itself.

5 Arithmetical Quanta and Qualia

Plotinus' theory of matter is mainly a recasting of Aristotle's theory in the platonist framework. Plotinus defined matter as the receptacle of the contingency and the possible, making it essentially undeterminate. His platonist constraints forces him to distinguish matter "there" which appears to be definable and intelligible, and matter "here" which has a sensible counterpart somehow related to the (discursive or not) soul(s). Now, by incompleteness, *any* possibility is, from the machine point of view, already something undeterminated. The logic G, for example, is closed for necessitation, and the logic G* $\setminus$ G is closed for possibilitation, i.e. if A is provable in G*, automatically the arithmetical possibility of A, i.e. the consistency of A, $\Diamond A$, i.e. $\neg\Box\neg A$, is provable in G* too, and is never provable in G. No formula with the shape $\Diamond\#$ is ever provable by G. Thus, a natural way to address the logic of certainty in this indeterminate frame will consist in defining a new intensional variant $\Box p \wedge \Diamond p$. This can be shown equivalent with $\Box p \wedge \Diamond t$, where t represents some classical tautology (or arithmetical "tautology" like "$0 = 0$"). Actually this can be generalized by using $\Diamond\Diamond t$ instead of $\Diamond t$, or even transfinitely by $\Diamond^{\alpha} t$ with α denoting a constructive ordinal, so that the material hypostases are infinite in number, but for the sake of simplicity I will treat only the simple case of $\Diamond t$. Using $\Diamond^{\alpha} t$ can be related to the autonomous progressions, see [20] for an introduction. Motivations which are not based on Plotinus, but based on thought experiments in mechanist philosophy of mind, can provide supplementary reasons to quantify the "material indetermination" by such new arithmetical connectors. The modal formula, where the box $\Box p$ is defined by $\Box p \wedge \Diamond t$, gives rise to a logic called Z ([41, 42]. Unlike S4Grz, due to the presence of the true but unprovable consistency $\Diamond t$, the logic Z, like the logics G, splits into a provable and unprovable but true parts, named naturally Z and Z*. The candidate for the arithmetical interpretation of the "intelligible matter" hypostases is the set of arithmetical interpretations of the logic Z (discursive, terrestrial) and Z* (true, divine).

To get the sensible matter hypostasis, which are more "soul-like", we have to reapply the theaetetical idea, and define again a new box by $(\Box A \wedge A) \wedge \Diamond t$. This gives again a *couple* of splitting logic X and X*.

To be sure the status of those logics is still rather mysterious, but we are not done yet : we must recall that we are not only interviewing a platonist universal, sufficiently introspective machine, but also a computationalist one, where the possible "mind states" have to belong to recursively enumerable sets of states (see section 3 above, see [44] for further motivations). This means we have to consider the intensional variants no more of G (and G*) but of V, or G1, that is G + "$p \rightarrow \Box p$" (see above). The corresponding logics will be denoted by the same name + "1", to remember that they are intensional variant of G1 = V. The "1" is a reminder of the computationalist Σ_1 restriction. So we get S4Grz1, which again is a non splitting logic, that is (cryptically) : S4Grz = S4Grz1*, and the splitted logics[17] Z1, Z1*, X1, X1*. The question we must address now is how close those logics are to a quantum logic of observation, as the empirical world seems to dictate. Let us define the modal logical system $\mathbf{B}^-$. It is the same system as the system $\mathbf{B}$ described above, except that the substitution rule is weakened so that in the formula $p \rightarrow \Box\Diamond p$, p can be substituted only by sentence letters, and also, $\mathbf{B}^-$ is not closed for the necessitation rule. We have the following theorem [41, 42] :

Theorem 4 : $\mathbf{B}^-$ provides a sound logical system for arithmetic (or any Löbian entity) by having its theorems proved in the three logics S4Grz1, Z1*, X1*.

This can be shown to be enough for defining three different arithmetical interpretations of some *quantum logic*. This is done in the Goldblatt way, where the quantization of atomic formula p are described by $\Box\Diamond p$ (see above, see also [57]), This, optimistically perhaps, could reflect the physicists doubts about which quantum logic would be correctly operating in nature [64, 58].

We have not yet been able to verify typical "quantum questions", like the question of (ortho)modularity, violation of Bell's inequality, etc. Some more technical considerations provides hints that a quantum "not" can be interpreted in one of those arithmetical "quantum logic" in a manner similar to Rawling and Selesnick [57]. Although all those propositionnal logic are decidable, by translating everything in G makes most such questions still intractable today. As far as we know, it seems to us that S4Grz1, Z1* and X1* provides the closer possible arithmetical interpretation of possible quantum logics[18]. Much work remains to dig on the significance of such, embryonic for sure, arithmetical and

[17] The logic Z, Z*, Z1, and Z1* have been recently axiomatized [66]. Unfortunately the technics used cannot be lifted to the X logics.

[18] We currently hope to get, from this arithmetical quantization, a sufficiently well behaved arithmetical projection operator, providing some relevant *Temperley Lieb algebra* [29] so as to justify the exploitability of a universal quantum computer in the vicinity of all Löbian

physical,in some neoplatonist sense, reality.

The main interest of such an arithmetical"introspective" approach relies in the fact that the gaps Z1* \ Z1 and X1* \ X1, provides kind of quantum logic for the non-communicable beliefs i.e. the non provable true statement corresponding to "physically" (in a sense close to Plotinus) self-inferable and measurable, like intensities, qualities, providing good candidates for "machine notion of qualia". If this happens to be correct, the quanta would appear to be definable as sharable (machine communicable) qualia. It should give a path from bits to qubits, with the advantage of justifying the communicable and uncommunicable assertion in the realm of what is observable. This could help to prevent mechanism from its common usual person elimination interpretation. An absence of justification of some universal quantum machine, from the lobian self-observing machine, or a mathematical proof that there are none, or any empirical difference between this arithmetical physics and the empirical physics, would refute, not Plotinus, but the present arithmetical interpretation.

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machine, relatively to their most probable computational histories, as seen from some first person plural person point of view, i.e. through some arithmetical version of the Intelligible Matter hypostasis. To be sure, the quantum entangling phenomenon remains hard to capture arithmetically, for the same reason that traditional quantum logic cannot represent the needed tensor product, but see [27] for some possible nuances on that question.

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