Belief function theory on the continuous space with an application to model based classification

Branko Ristic
ISR Division, DSTO
PO Box 1500, Edinburgh SA 5111
Australia
(in IRIDIA during 2003/04)
branko@iridia.ulb.ac.be

Philippe Smets
IRIDIA, Université libre de Bruxelles
50 Av. Roosevelt, CP 194-6, 1050 Bruxelles
Belgium
psmets@ulb.ac.be
http://iridia.ulb.ac.be/˜psmets

Abstract

The paper defines belief functions on continuous frames of discernment, where masses generalize into densities. Explicit and manageable solutions can be formulated when densities are only assigned to the intervals of $\mathbb{R}$. When our domain knowledge is represented by the pignistic probability density, then we build the corresponding least committed belief function. The theory is applied to model based classification and the results are compared to the classical Bayesian approach.

Keywords: Belief function theory, evidential theory, transferrable belief model, target classification.

1 Introduction

The belief function theory (evidential theory) has been primarily developed for discrete frames of discernment (frames). Following [9],[15], this paper defines belief functions on continuous frames, where belief masses generalize into belief densities. Explicit and manageable solutions can be formulated when densities are assigned only to the intervals on the real axis $\mathbb{R}$, although the theory is conceptually valid for $\mathbb{R}^n$.

When our domain knowledge is partial and represented only by a potential betting behavior on the observation (in the continuous domain), we model it by a pignistic probability density. In this case we can build the least committed belief function among those which correspond to the given one. Then we can apply the usual tools of the belief function theory, such as the Generalised Bayesian theorem, combination rules (e.g. Dempster's rule of combination), etc. The theory is applied to model based target classification and the results are compared to those achieved by the classical Bayesian approach.

We accept that beliefs are quantified by belief functions as described in the transferable belief model (TBM) [14]. Classical material about belief functions and the TBM can be found in Shafer [8] and Smets [11]. In order to simplify our presentation we will first consider the case where the frame is the interval $[0, 1] \subset \mathbb{R}$, and there is only a final number of focal sets. Later this will be relaxed to the continuous domain (with an infinite number of focal sets) and the entire real axis $\mathbb{R}$.

2 Belief functions on $\mathbb{R}$

This section presents the extracts from a more thorough study presented in [13]. Consider a nonempty interval on the real axis $\mathbb{R}$, denoted as $[\alpha, \beta] \subseteq \mathbb{R}$, $\alpha < \beta$. Let $\mathcal{I}_{[\alpha,\beta]}$ be a set of closed intervals in $[\alpha, \beta]$. Formally,

$$\mathcal{I}_{[\alpha,\beta]} = \{[x, y] : x \geq \alpha, x \leq y \leq \beta\}.$$

We assume that masses are only alloccated to closed intervals. It implies that for a collection of pairwise disjoint intervals in $\mathcal{I}_{[\alpha,\beta]}$, the belief functions satisfy a special form of additivity. Formally, $\forall A_1, A_2, \ldots, \in \mathcal{I}_{[\alpha,\beta]}$, such that

$$A_{i_1} \cap A_{i_2} = \emptyset,\ i_1, i_2 \in \{1, 2, \ldots\}, i_1 \neq i_2$$

we have:

$$bel^{\mathcal{I}_{[\alpha,\beta]}}\Big(\bigcup_{i=1,2,\ldots} A_i\Big) = \sum_{i=1,2,\ldots} bel^{\mathcal{I}_{[\alpha,\beta]}}(A_i)$$

2.1 Finite number of focal sets

Let $\mathcal{A}$ be a subset of $\mathcal{I}_{[0,1]}$ consisting of a *finite* number of non-empty intervals on $[0, 1]$:

$$\mathcal{A} = \{A_i : A_i \in \mathcal{I}_{[0,1]}; i = 1 \ldots, n\} \cup \{\emptyset\}$$

For convenience, use notation $A_0 = \emptyset$. Function $m^{\mathcal{A}} : \mathcal{A} \to [0,1]$ is a basic belief assignment (bba) with the property $\sum_{i=0}^{n} m^{\mathcal{A}}(A_i) = 1$. The A_i's with $m^{\mathcal{A}}(A_i) > 0$ are the focal sets of this bba.

There is a very convenient graphical representation of these intervals: every $A = [a,b]$, such that $a, b \in [0,1]$ and $a \leq b$, corresponds to a single point in the triangle of Figure 1, and vice versa. This triangle is defined as:

$$\mathcal{T}_{[0,1]} = \{(x,y) : x, y \in [0,1], x \leq y\}.$$

To each point in the triangle $\mathcal{T}_{[0,1]}$ that corresponds to a focal set of $m^{\mathcal{A}}$, we assign a mass equal to the basic belief mass. Hence $m^{\mathcal{A}}([a,b])$ is assigned to the point $(a,b) \in \mathcal{T}_{[0,1]}$ for every $A \in \mathcal{A}$. When $m^{\mathcal{A}}(\emptyset) = 0$, the result of this assignment is a (discrete) probability distribution function on $\mathcal{T}_{[0,1]}$, i.e. $P\{(x,y) = (a,b)\} = m^{\mathcal{A}}([a,b])$. The convention for axes x and y is adopted as shown in Figure 1. In order to further illustrate this concept, consider the following example.

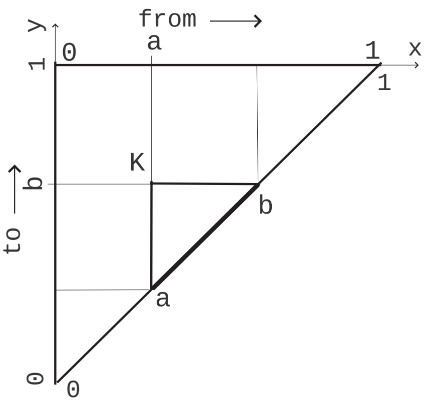


Figure 1: *Point $K = (a,b)$ inside the triangle $\mathcal{T}_{[0,1]}$, uniquely defines the interval $[a,b] \subseteq [0,1]$*

Example 1. Table 1 defines a bba with six focal sets, depicted in Figure 2 inside the triangle $\mathcal{T}_{[0,1]}$. Let $A = [a,b]$ be an interval in $[0,1]$, with $a = 0.2$ and $b = 0.7$. Let us now work out the belief, the commonality and the plausibility functions ($bel^{\mathcal{A}}$, $q^{\mathcal{A}}$ and $pl^{\mathcal{A}}$, respectively) of interval A. The sign $\times$ in last three columns of Table 1 indicate the masses to be included in $bel^{\mathcal{A}}(A)$, $q^{\mathcal{A}}(A)$ and $pl^{\mathcal{A}}(A)$.

$bel^{\mathcal{A}}(A), A = [a,b]$, is the sum of all the masses given to the subsets of A, thus to the non-empty intervals $A_i = [a_i, b_i]$, where $[a_i, b_i] \subseteq [a,b]$, i.e. $a_i \geq a$ and $b_i \leq b$. Graphically, every mass included in $bel^{\mathcal{A}}(A)$ must lie in the shaded triangle of Figure 3.(a) – this triangle contains all (and only) the intervals $[x,y]$ such that $x \geq a$ and $y \leq b$. Hence, to compute $bel^{\mathcal{A}}(A)$ one adds up the masses of all the focal sets located in this triangle. In our example $bel^{\mathcal{A}}(A) = 0.12$.

Table 1: *bba defined on $\mathcal{A}$ with six focal sets, and the corresponding belief, commonality and plausibility of $A = [0.2, 0.7]$*

		$A_i =$	$[a_i, b_i]$			
i	$m^{\mathcal{A}}$	a_i	b_i	$bel^{\mathcal{A}}$	$q^{\mathcal{A}}$	$pl^{\mathcal{A}}$
1	.07	.3	.4	$\times$		$\times$
2	.18	.1	.9		$\times$	$\times$
3	.25	.1	.8		$\times$	$\times$
4	.15	.4	.9			$\times$
5	.05	.4	.5	$\times$		$\times$
6	.30	.8	.9			
total	1.			.12	.43	.70

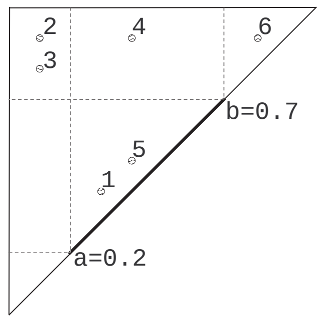


Figure 2: *Graphical representation of the focal set corresponding to Table 1*

$q^{\mathcal{A}}(A), A = [a,b]$, is defined as the sum of the masses given to the intervals $A_i = [a_i, b_i]$, where $[a,b] \subseteq [a_i, b_i]$, i.e. $a_i \leq a$ and $b_i \geq b$. Graphically, every mass included in $q^{\mathcal{A}}(A)$ must lie in the shaded rectangle of Figure 3.(b) – this rectangle contains all (and only) the intervals $[x,y]$ such that $x \leq a$ and $y \geq b$. Hence to compute $q^{\mathcal{A}}(A)$ one adds up the masses of the focal sets located inside this shaded rectangle. In our example, $q^{\mathcal{A}}(A) = 0.43$.

$pl^{\mathcal{A}}(A), A = [a,b]$, is defined as the sum of the masses given to the intervals $A_i = [a_i, b_i]$, where $[a_i, b_i] \cap [a,b] \neq \emptyset$, i.e. $a \leq b_i$ and $b \geq a_i$. Graphically, every mass included in $pl^{\mathcal{A}}(A)$ must lie in the shaded area of Figure 3.(c) – this area contains all (and only) the intervals $[x,y]$ such that $x \leq b$ and $y \geq a$. Hence to compute $pl^{\mathcal{A}}(A)$ one adds up the masses of the focal sets located inside this shaded area. In our example, $pl^{\mathcal{A}}(A) = 0.70$.

The singletons of $\mathcal{I}_{[0,1]}$ are zero-length intervals. If all the focal sets are non-empty intervals (as in our example), we can compute the pignistic probability density function (pdf) over singletons s (where

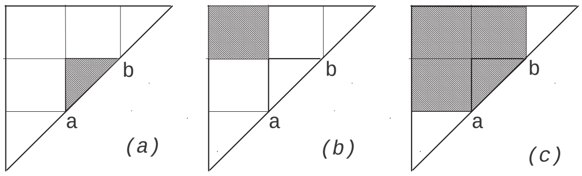


Figure 3: *Graphical representation of (a) belief; (b) commonality; (c) plausibility*

$0 \leq s \leq 1$) as follows [11]:

$$Betf(s) = \sum_{A: s \in A \subset [0,1]} \frac{m^{\mathcal{A}}(A)}{|a^* - a_*| \, [1 - m^{\mathcal{A}}(\emptyset)]}$$

where $a_* = \inf\{a : a \in A\}$ and $a^* = \sup\{a : a \in A\}$. The computation of pignistic pdf involves the focal sets located in the rectangular area of triangle $\mathcal{T}_{[0,1]}$ defined by $0 \leq x \leq s$ and $s \leq y \leq 1$. In our example, the pignistic pdf of say $s = 0.35$ would involve the focal sets $1, 2$ and 3 and would result in: $Betf(0.35) = \frac{0.07}{0.1} + \frac{0.18}{0.8} + \frac{0.25}{0.7} = 1.28$. $Betf$ is a proper probability density function.

2.2 Continuous domain

Next we relax the assumption that the number of focal elements is finite. The finite collection of subintervals $\mathcal{A}$, is now replaced by $\mathcal{I}_{[0,1]}$. Instead of discrete probabilities defined on the triangle $\mathcal{T}_{[0,1]}$, we now assign a probability density over the entire area of $\mathcal{T}_{[0,1]}$. What we described so far essentially will remain valid, except that masses become densities and sums become integrals.

Let $m([a, b])$ be a basic belief density (bbd) (we replace bbm by bbd to enhance that m is now a density). Let $f^{\mathcal{T}_{[0,1]}}(a, b) = m([a, b])$. Then $f^{\mathcal{T}_{[0,1]}}$ is a density function on $\mathcal{T}_{[0,1]}$: $f^{\mathcal{T}_{[0,1]}} : \mathcal{T}_{[0,1]} \to [0, \infty)$ with the property that:

$$\int_{x=0}^{x=1} \int_{y=x}^{y=1} f^{\mathcal{T}_{[0,1]}}(x, y) \, dx \, dy = 1 \qquad (1)$$

Normalisation of the basic belief density as in (1) in fact is not necessary. The integral of $f^{\mathcal{T}_{[0,1]}}$ over $\mathcal{T}_{[0,1]}$ may be allowed to result in a value that is less than 1, with the missing belief allocated to the empty set, just as it was done in TBM [11].

Let us now define $bel^{\mathcal{I}_{[0,1]}}$, $pl^{\mathcal{I}_{[0,1]}}$ and $q^{\mathcal{I}_{[0,1]}}$ functions corresponding to $f^{\mathcal{T}_{[0,1]}}$. According to the explanations given so far, these functions will be the integrals of $f^{\mathcal{T}_{[0,1]}}$, with the limits of integration defined by the shaded areas in Figure 3. Thus we have:

$$\begin{aligned}
bel^{\mathcal{I}_{[0,1]}}([a, b]) &= \int_{x=a}^{x=b} \int_{y=x}^{y=b} f^{\mathcal{T}_{[0,1]}}(x, y) \, dx \, dy \\
pl^{\mathcal{I}_{[0,1]}}([a, b]) &= \int_{x=0}^{x=b} \int_{y=\max(a,x)}^{y=1} f^{\mathcal{T}_{[0,1]}}(x, y) \, dx \, dy \\
q^{\mathcal{I}_{[0,1]}}([a, b]) &= \int_{x=0}^{x=a} \int_{y=b}^{y=1} f^{\mathcal{T}_{[0,1]}}(x, y) \, dx \, dy
\end{aligned}$$

Using derivative-integral identities one can also write:

$$f^{\mathcal{T}_{[0,1]}}(a, b) = -\frac{\partial^2 \, bel^{\mathcal{I}_{[0,1]}}([a, b])}{\partial a \, \partial b} \qquad (2)$$

$$f^{\mathcal{T}_{[0,1]}}(a, b) = -\frac{\partial^2 \, q^{\mathcal{I}_{[0,1]}}([a, b])}{\partial a \, \partial b} \qquad (3)$$

The pignistic density function $Betf$ can be derived from $f^{\mathcal{T}_{[0,1]}}(x, y)$ as follows:

$$Betf(a) = \lim_{\epsilon \to 0} \int_{x=0}^{x=a} \int_{y=a+\epsilon}^{y=1} \frac{f^{\mathcal{T}_{[0,1]}}(x, y)}{y - x} \, dx \, dy, \qquad (4)$$

for $a \in [0, 1]$. We do not put directly $\epsilon = 0$ in (4) in order to avoid division by zero.

Example 2. Consider the uniform density on $\mathcal{T}_{[0,1]}$, that is

$$f^{\mathcal{T}_{[0,1]}} = 2, \qquad \forall x, y \in [0, 1], x \leq y$$

Then using (4) we get

$$Betf(a) = -2[(1 - a)\log(1 - a) + a \log a]$$

for $0 < a < 1$ (see Figure 4).

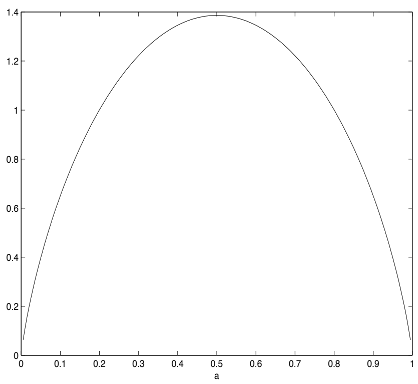


Figure 4: *$Betf(a)$ generated by a uniform density on $\mathcal{T}_{[0,1]}$*

Generalisation to $\mathbb{R}$. So far we have developed belief functions on $\mathcal{I}_{[0,1]}$, as it simplifies the presentation and allows for nice graphical representation in $[0, 1]$ interval. However, all concepts directly apply when the frame of discernment is the entire real axis $\mathbb{R}$; one just needs to replace 0 (the lower limit) with $-\infty$ and 1 (the upper limit) with $+\infty$. Thus $[0,1]$ is replaced with $(-\infty,\infty)$. Let us denote by $\mathcal{I}$ the set of intervals on the real axis $\mathbb{R}$ and by $\mathcal{T}$ the set of pairs $(x,y) \in \mathbb{R}^2 : x \leq y$. Then we say that m, bel, q and pl are defined on the Borel sigma algebra generated by $\mathcal{I}$ and f is defined on $\mathcal{T}$.

3 The least committed bbd

Suppose your domain knowledge is partial and based only on some potential betting behaviour, represented by the pignistic density function $Betf(a)$. Since the pignistic transform is many-to-one transform, an infinite number of belief density functions can induce the same $Betf$. These belief funtions are said to be isopignistic. In order to apply the belief function theory (in the continuous domain) one needs to formulate a method of building a belief density (BD) from the pignistic density. The least commitment principle [11],[5] suggests to choose among all iso-pignistic belief densities, the belief density which maximizes the commonality function q. As in the discrete case [12], the q least committed belief density is a consonant belief density. On the real axis $\mathbb{R}$ this means that all focal sets on $\mathcal{I}$ are nested, i.e. can be ordered in such a way that each focal interval is contained by the following one.

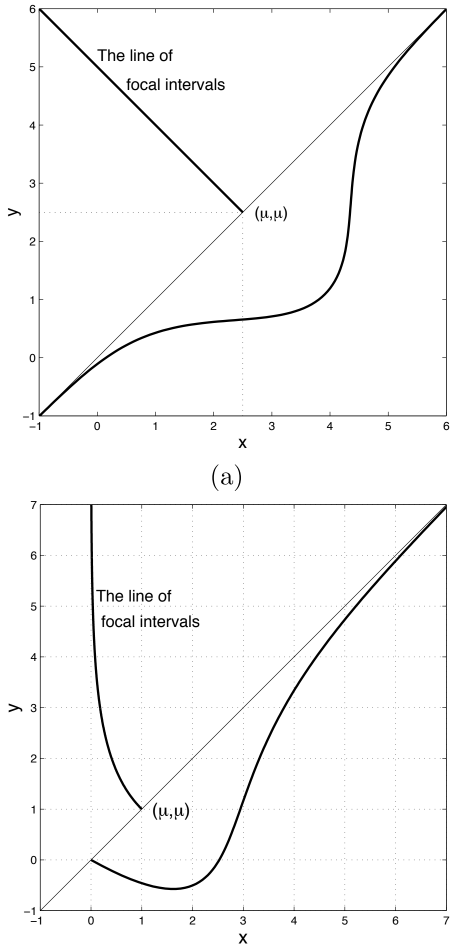
We will further concentrate on a unimodal pignistic density with a mode $\mu = \arg\max_a Betf(a)$. The focal sets of the least committed (LC) belief density are intervals $[a,b]$ which satisfy: $Betf(a) = Betf(b)$. Consequently, for every focal interval of the LC-BD, $[a,b]$, we have that $\mu \in [a,b]$. Another very important property of the focal intervals of the LC-BD is that they form a line on the triangle $\mathcal{T}$. This line has the following properties:

- It starts from $(x,y) = (\mu,\mu)$; the plausibility at this point is $pl^{\mathcal{I}}([\mu,\mu]) = 1$.
- For all symmetrical pignistic densities $Betf$ (e.g. normal, Laplace, Cauchy), centered at μ, this is a straight line given by:

$$y = -(x - 2\mu) \qquad -\infty < x \leq \mu$$

Figure 5 shows the line of focal intervals in $\mathcal{T}$ for (a) normal pignistic density with $\mu = 2.5$ and $\sigma = 1$; (b) gamma pignistic density $Betf(s) = s\, e^{-s}$, $(s > 0)$, with the mode $\mu = 1$.

The relationship between $Betf$ and any basic belief density in general is expressed by (4). Let us



(a)

(b)

Figure 5: *The focal sets of the LC belief density (solid line in the upper triangle) induced by: (a) normal pignistic density; (b) gamma pignistic density*

denote the LC bbd (induced by $Bel\,f$) as $\varphi(u)$ where $u \geq 0$. We have seen that the focal sets of this bbd are points on a line in $\mathcal{T}$, and u corresponds to the distance from the point (μ,μ). Due to this specific form of the LC bbd, the relationship between $Betf(s)$ and $\varphi(u)$ has a much simpler form than in (4). If $s \geq \mu$, then

$$Betf(s) = \int_{u=s}^{\infty} \frac{\varphi(u)}{u - \bar{u}} du \qquad (5)$$

where $\bar{u}$ is defined by the property that: $Betf(\bar{u}) = Betf(u)$. Note that $\bar{u}$ is a function of u. By differentiation of (5) we obtain that:

$$\varphi(s) = -(s - \bar{s}) \frac{Betf(s)}{ds} \qquad (6)$$

The bbd $\varphi(s)$ is always positive since:

(1) $s \geq \bar{s}$ and

(2) $\frac{dBetf(s)}{ds} < 0$ for $s \geq \mu$.

For model based classification problems, we apply the generalised Bayes theorem [11] which requires to compute the plausibility function from the bbd. Since the LC bbd is a consonant belief function, with the property that its focal sets are the points along a line in $\mathcal{T}$, we can write:

$$\begin{aligned} pl(x) &= \int_x^\infty \varphi(a)\, da & (7) \\ &= -\int_x^\infty (a-\bar{a})\ [Betf(a)]'\ da & (8) \end{aligned}$$

The limits of integration in (7) reflect the fact that only the focal intervals with the property $x \leq a \leq \infty$ will have a non-empty intersection with x. Using the differentiation rule: $(uv)' = uv' + uv'$ and the property of our unimodal bbd: $\lim_{x\to\infty} Betf(x) = 0$ we obtain:

$$\begin{aligned} pl(x) =&(x-\bar{x})\, Betf(x) \\ &+ \int_x^\infty \left(1 - \frac{d\bar{a}}{da}\right)\ Betf(a)\ da \quad (9) \end{aligned}$$

Example 3. Suppose the pignistic density is a normal density, i.e. $Betf(x) = \mathcal{N}(x;\mu,\sigma)$. In order to work out the LC bbd $\varphi(x)$ and its corresponding plausibility $pl(x)$ we make the standard substitution $y = (x-\mu)/\sigma$. In this case $y - \bar{y} = 2y$ and thus $d\bar{y}/dy = -1$. Application of (6) and (9) yields for $y \geq 0$:

$$\begin{aligned} \varphi(y) &= \frac{2y^2}{\sqrt{2\pi}} e^{-y^2/2} & (10) \\ pl(y) &= \frac{2y}{\sqrt{2\pi}} e^{-y^2/2} + \mathrm{erfc}(y/\sqrt{2}) & (11) \end{aligned}$$

where $\mathrm{erfc}(s) = \frac{2}{\sqrt{\pi}} \int_s^\infty e^{-t^2} dt$. It follows than: $\varphi(x) = \varphi(y)/\sigma$ and $pl(x) = pl(y)$. The two functions are shown in Figure 6 for $\mu = 1$ and $\sigma = 1.5$.

Example 4. Let $Betf(x)$ be an exponential density:

$$Betf(x) = \begin{cases} \frac{1}{\theta} e^{-(x-a)/\theta} & x \geq a \\ 0 & x < a \end{cases} \quad (12)$$

Using the substitution $y = (x-a)/\theta$ we note that the LC bbd is a Gamma density: $\varphi(y) = ye^{-y}$ for $y \geq 0$. The plausibility is then $pl(y) = (1+y)e^{-y}$. As before, $\varphi(x) = \varphi(y)/\theta$ and $pl(x) = pl(y)$.

4 Application to model-based target classification

In order to demonstrate an application of the theory presented above, let us consider one of the most difficult problems in military air surveillance: correct identification of non-cooperative flying objects in the surveillance volume. In general three groups of target attributes (features) are exploited for identification, those based on target shape, kinematic behaviour and electro-magnetic (EM) emissions [1]. Let us consider a simple example where the aim is to classify targets into one of the three platform categories [7]:

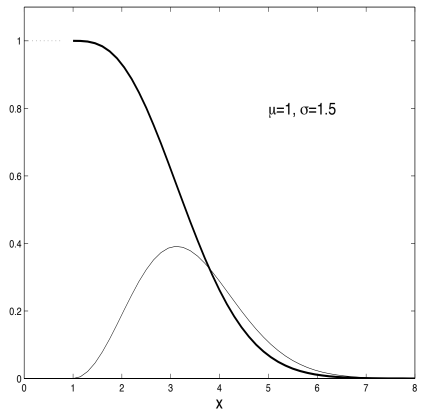


Figure 6: *The LC bbd $\varphi(x)$ (thin line) and its plausibility $pl(x)$ (thick line), corresponding to $Betf(x) = \mathcal{N}(x;1,1.5)$*

Class 1 - Commercial planes;

Class 2 - Large military aircrafts (such as transporters, bombers);

Class 3 - Light and agile military aircrafts (fighter planes).

4.1 Speed as a target feature

We will assume that the only available target feature is its speed (a kinematic feature obtained from the radar) [2], [16]. The speed profiles for our three classes can be described by Table 2 [2]:

Table 2: Speed intervals for three air platform categories (in km/h)

Target class	Min	Max
Commercial (c_1)	560	885
Bomber (c_2)	400	725
Fighter (c_3)	525	950

First we present target classification using the Bayesian classifier, which is followed by the Belief function classifier.

Bayesian analysis. In order to apply the Bayesian classifier we must adopt a suitable probability density function of the speed conditioned on the class. Various possibilities are applicable, such the uniform, beta, Gaussian, etc. Let us adopt the Gaussian densities, with the parameters selected in such a way that $P\{s_{\min} < x < s_{\max}\} = 0.99876$, where $[s_{\min}, s_{\max}]$ is the speed interval given in Table 2. Figure 7 shows the distribution of speed feature conditioned on the class. The hypothesis space is defined as $C = \{c_1, c_2, c_3\}$. The Bayesian classifier (assuming the uniform prior for classes) will compute the probability of class c_i $(i = 1, 2, 3)$ given feature x as:

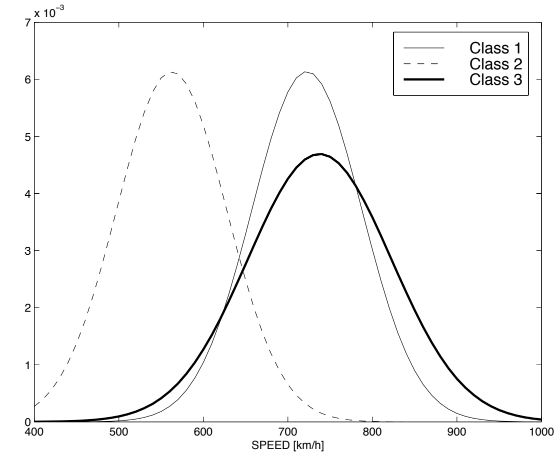


Figure 7: *Adopted pdf models (Gaussian) of target speed, conditioned on the class*

$$P\{c_i|x\} = \alpha\, p(x|c_i) \qquad (i = 1, 2, 3) \qquad (13)$$

where α is a normalisation constant. Figure 8 displays the class probabilities $P\{c_i|x\}$ computed for a range of speed values $x \in [400, 1000]$ km/h.

Belief function analysis. The belief function analysis can start from the very same pdf models adopted for the Bayesian analysis (Figure 7). However, their meaning is different. Since our probabilistic knowledge is very scarce and incomplete (we just know the speed limits for each target class) these models are now considered as pignistic densities of speed x conditioned on class c_i, denoted as $Betf(x|c_i)$. The first step is to build the least committed belief function over the observation space which corresponds to $Betf(x|c_i)$, followed by the application of the Generalised Bayesian Theorem (GBT) [10], [3]. The key here is to compute likelihoods $l(c_i|x) = pl(x|c_i)$, which is done using equation (9). Then the GBT yields for every subset $A \subseteq C$ the following bba:

$$m(A|x) = \prod_{c_i \in A} pl(x|c_i) \prod_{c_i \in \bar{A}} [1 - pl(x|c_i)] \qquad (14)$$

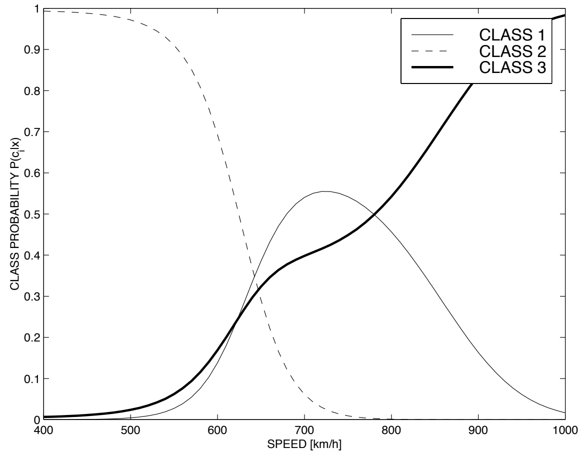


Figure 8: *Bayesian analysis: class probabilities conditioned on target speed x*

Finally the last step is to apply the pignistic transform to $m(A|x)$ to compute the pignistic class probabilities:

$$BetP\{c_i|x\} = \sum_{A\,:\, c_i \in A} \frac{1}{|A|} \frac{m(A|x)}{[1 - m(\emptyset|x)]} \qquad (15)$$

The resulting pignistic class probabilities are shown in Figure 9, for a range of speed values $x \in [400, 1000]$ km/h.

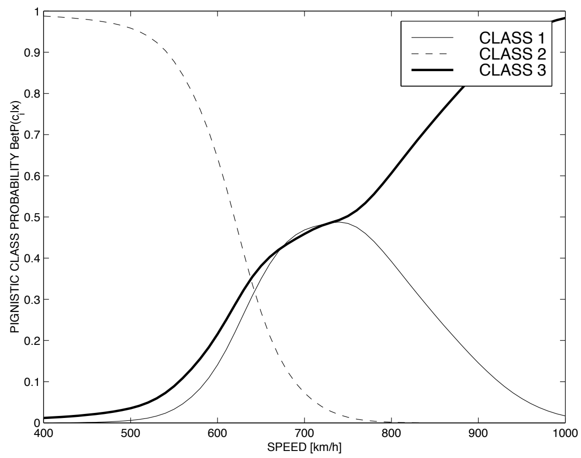


Figure 9: *Belief function analysis: pignistic class probabilities conditioned on target speed x*

Comparing Figures 8 and 9 one can observe similar performance of both classifiers for speeds less than 650 km/h and greater than 770 km/h. However, in the range [650, 770] km/h, where the pdf of class 1 and 3 overlap, the Bayesian classifier favours class 1, while the Belief classifier is undecided between 1 and 3. We argue that being undecided makes more sense, because the most likely observation of speed, for both class 1 and 3, falls in this region. A similar effect will be illustrated and discussed in the next subsection.

4.2 Acceleration as a target feature

Suppose the only available target feature is its maximum acceleration, denote as a (usually observed during a certain interval of time). Target acceleration can be useful [6] because it is related to target maneuverability. For class 1, the acceleration is rarely higher than $1g$ (where $g = 9.81$ m/s^2 is the gravitation due to gravity), because the acceleration higher than $\pm 1g$ causes sickness in passengers. Targets of class 2 sometimes perform mild evasive manoeuvres but their maximum acceleration (due to their size) is rarely higher than $\pm 4g$. Targets of class 3 are light and agile, with highly trained pilots - the maximum acceleration of modern fighter planes can go up to $\pm 7g$. The steady-state of acceleration, however, for all three classes of targets is *zero*. This is so because a constant velocity flight (i.e. with zero acceleration) ensures the minimum fuel consumption and the least stress for a pilot.

As before, we model the pdf of the feature (maximum acceleration) conditioned on the class, $p(a|c_i)$, $i = 1, 2, 3$. A reasonable model is a zero-mean Gaussian density with three different values of standard deviation, as shown in Figure 10 (the unit of target acceleration is g). Standard deviations of Gaussian densities are adopted as follows: $\sigma_1 = 0.4g$, $\sigma_2 = 1.6g$ and $\sigma_3 = 2.8g$. These values are selected to ensure that $P\{|a| < \gamma\} = 0.99876$, where $\gamma = 1g$,$4g$ and $7g$ for class 1, 2 and 3 respectively.

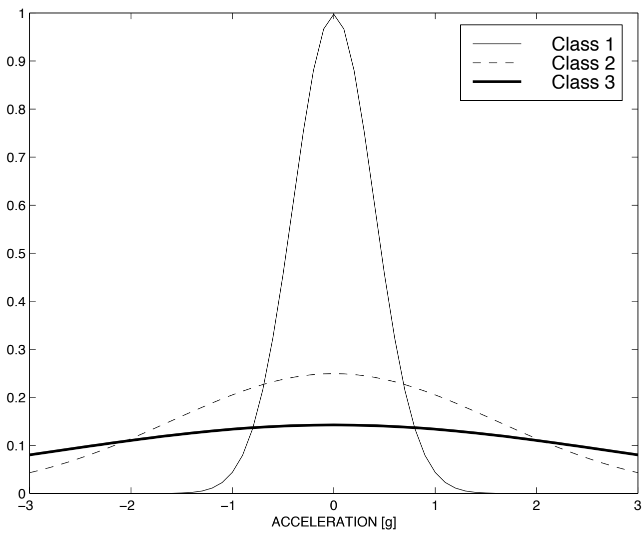


Figure 10: *Adopted pdf models of target acceleration conditioned on the class*

The classification results (for acceleration values in the interval $[-3g, 3g]$) are shown in Figures 11 and 12 for the Bayesian and the belief function classifier, respectively. Observe that for small accelerations ($|a| < 0.25g$), the Bayesian classifier decisively declares a target to be of class 1, while the belief function classifier does not favour any class. Again we argue that for the available prior knowledge about targets and accelerations, being undecided for small accelerations makes more sense: small accelerations are not a distinguishing feature between the classes.

The crux of the belief function analysis is that the LC plausibility $pl(a|c_i)$ is close to 1 if acceleration a approximately equals the mode of $p(a|c_i)$ (see Section 3). Interestingly, a somewhat similar classification result can be obtained in this example using an ad-hoc fudge in the Bayesian classifier, by replacing densities $p(a|c_i)$ with $p(a|c_i)/\max\{p(a|c_i)\}$ [4]. The proposed framework of belief function theory (the least commitment principle, generalised Bayesian theorem, pignistic transform), however, provides a sound theoretical basis for target classification without a need for any fudge.

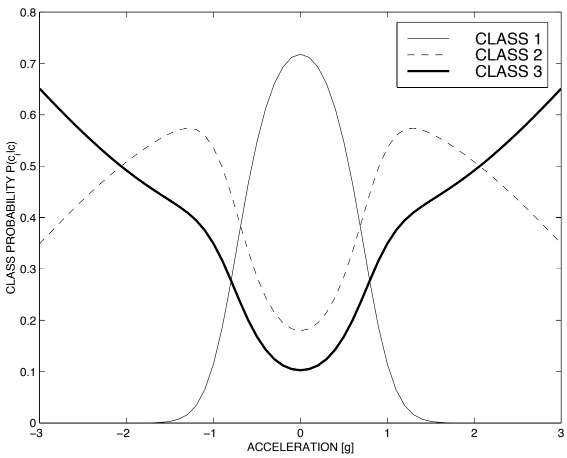


Figure 11: *Bayesian analysis: class probabilities conditioned on acceleration*

5 Conclusions

The paper presents a theoretical framework of the belief function theory in the continuous domain, where the frame of discernment is the real axis $\mathbb{R}$ (or its segment). When the probabilistic description of observations in the continuous domain is incomplete, we represent it by the pignistic probability density. When the pignistic density is unimodal, the focal sets of the least committed belief function which corresponds to this density, form

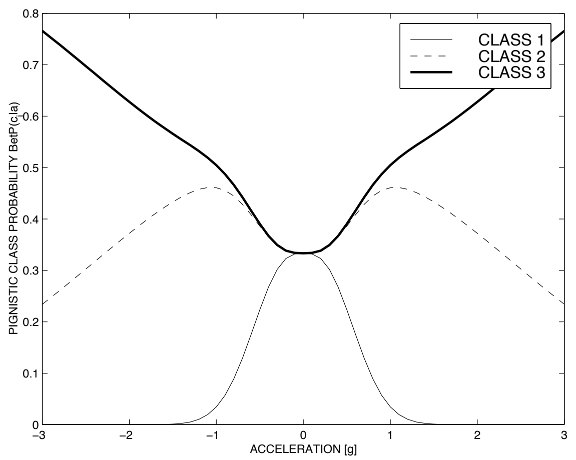


Figure 12: *Belief function analysis: class probabilities conditioned on acceleration*

a line in $\mathbb{R}^2$. This greatly simplifies the relationships between the basic belief density, pignistic density and the plausibility function. The theory has been applied to the model-based target classification, where observations of target speed and acceleration (in the continuous domain) are used as a feature. The classifier based on the belief function theory appears to be very simple to implement and produces results which are arguably more meaningful than those obtained using the Bayesian classifier.

For n-dimensional measurement space (e.g. speed and acceleration considered as a joint measurement) we would need to extend the theory to the case where the frame is $\mathbb{R}^n$. If features are independent, extending the theory to $\mathbb{R}^n$ would be manageable. If features are not independent, a transformation into new independent features would be first required.

References

[1] S. Blackman and R. Popoli. *Design and Analysis of Modern Tracking Systems.* Artech House, 1999.

[2] A. Caromicoli and T. Kurien. Multitarget identification in airborne surveillance. In *Proc. SPIE*, volume 1098, pages 161–176, 1989.

[3] F. Delmotte and P. Smets. Target identification based on the transferable belief model interpretation of Dempster-Shafer model. *IEEE Trans. SMC.* In review.

[4] T. Denoeux. Private communication. 2003.

[5] D. Dubois and H. Prade. The principle of minimum specificity as a basis for evidential reasoning. In B. Bouchon and R. Yager, editors, *Uncertainty in knowledge-based systems*, pages 75–84. Springer Verlag, 1987.

[6] H. Leung and J. Wu. Bayesian and Dempster-Shafer target identification for radard surveillance. *IEEE Trans. Aerospace and Electronic Systems*, 36(2):432–447, 2000.

[7] B. Ristic, N. Gordon, and A. Bessell. On target classification using kinematic data. *Information Fusion*, 2003. In Press.

[8] G. Shafer. *A mathematic theory of evidence.* Princeton University Press, 1976.

[9] P. Smets. *Un modéle mathématico-statistique simulant le processus du diagnostic médical.* PhD thesis, Université libre de Bruxelles, 1978. available through University Microfilm International, 30-32 Mortimer street, London, W1N 7RA, thesis 80-70, 003.

[10] P. Smets. Belief functions: the disjunctive rule of combination and the generalized Bayesian theorem. *International Journal of Approximate Reasoning*, 9:1–35, 1993.

[11] P. Smets. The transferable belief model for quantified belief representation. In P. Smets, editor, *Quantified Representation of Uncertainty & Imprecission*, volume 1 of *Handbook of Defeasible Reasoning and Uncertainty Mangement Systems*, pages 267–301. Kluwer, Doordrecht, 1998.

[12] P. Smets. Quantified epistemic possibility theory seen as a hyper cautious transferable belief model. In *Rencontres Francophones sur la Logique Floue et ses Applications*, pages 343–353. Cépaduès-Éditions, Oct. 2000. http://www.editions-cepadues.fr.

[13] P. Smets. Belief functions on continuous spaces. *International Journal of Approximate Reasoning*, 2004. (Submitted).

[14] P. Smets and R. Kennes. The transferable belief model. *Artificial Intelligence*, 66:191–234, 1994.

[15] T. M. Strat. Continuous belief functions for evidential reasoning. In *Proc. National Conf. Artificial Intelligence*, pages 308–313. Amer. Assoc. Artif. Intelligence (AAAI), Aug. 1984.

[16] J. G. Teti, R. P. Gorman, and W. A. Berger. A multifeature decision space approach to radar target identification. *IEEE Aerospace and Electronic Systems*, 32(1):480–487, 1996.