QUANTIFYING BELIEFS BY BELIEF FUNCTIONS: AN AXIOMATIC JUSTIFICATION.

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Summary: We present a set of axioms that justify the use of belief functions to quantify the beliefs held by an agent Y at time t and based on Y's evidential corpus. It is essentially postulated that degrees of belief are quantified by a function in [0,1] that give the same degrees of beliefs to subsets that represent the same propositions according to Y's evidential corpus. We derive the impact of the coarsening and the refinement of the frame on which the beliefs are expressed. The conditioning process is also derived. We propose a closure axiom that asserts that any measure of beliefs can be derived from other measures of beliefs defined on less specific frames.

Keywords: Belief functions, quantified beliefs, subjective probabilities, axioms for beliefs, transferable belief model.

1. Introduction.

Uncertainty induces beliefs[2], i.e. dispositions that guide our behaviour. It sounds natural to try and quantify them on a numerical scale. These quantified beliefs manifest themselves at two levels: the **credal level** where beliefs are entertained and the **pignistic level** where beliefs are used to take decisions (pignus = a bet in Latin, Smith 1961). Usually these two levels are not distinguished and probability functions are used to quantify beliefs at both levels. The justification is usually linked to "rational" agent behaviour within betting and decision contexts (DeGroot 1970). The Bayesians have convincingly showed that if decisions must be "coherent", our beliefs over the various possible outcomes must be quantified by a probability function. This result is accepted here, except that such *probability functions quantify our beliefs only when a decision is really involved*. That beliefs are necessary ingredients for our decisions does not mean that beliefs cannot be entertained without any revealing behaviour manifestations (Smith and Jones, p.147).

In this paper, we present a set of axioms that must be satisfied by the function that should be used to quantify the beliefs held at the credal level. We call that function a credibility function.

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[2] A belief is a proposition which you could doubt. Here, it is endowed by a strength.

It will be shown that the credibility function is a belief function. The resulting model is the transferable belief model (Smets and Kennes, 1990, Smets, 1990a, Smets, 1988)

We accept all over that degrees of beliefs at both the credal and the pignistic levels are pointwise defined, degrees of beliefs satisfying a total order.

All beliefs entertained by an agent Y at time t and their degrees are defined relative to a given **evidential corpus** (EC_t^Y) i.e., the set of pieces of evidence in Y's mind at time t. Our approach is normative, the agent Y is an ideal rational agent, the evidential corpus is deductively closed and it induces unique degrees of belief. One source of modification in EC_t^Y is **updating**: it results from the adjunction to the corpus of a new piece of evidence assumed to be true and compatible with EC_t^Y. The updating is similar to the expansion process considered in Gardenfors (1988). Only one agent Y is considered in this paper, and time t is unique except when updating will be studied.

This paper summarizes the major results. Details and proofs are presented in Smets (1992b). We present successively the propositional space on which credibility functions ared defined (section 2), the principle axioms characterizing the credibility functions (section 3), the dynamic of the credibility functions after non-informative coarsening (section 4) and refinement (section 5) of the frame of discernment and adfter updating of the evidential corpus by an expansion process (section 6). A closure property is presented that implies that credibility functions are belief functions (section 7).

Lengthy discussions about the use and appropriateness of the belief functions to quantify beliefs can be found in two special issues of the International Journal of Approximate Reasoning (volumes 4(5):1990 and 6(3):1992). These problems are not tackled here. We only try to find axioms that justify the use of belief functions for quantifying beliefs.

2. The propositional space.

This section defines the domain on which the agent Y will express his beliefs at time t. These beliefs are quantified by a function Cr that we derive in this paper.

Our presentation is based on **possible worlds** (Carnap, 1956, 1962, Ruspini, 1986, Bradley and Swartz, 1979). Let L be a finite **propositional language**. Let $\Omega = \{\omega_1, \omega_2, ...\omega_n\}$ be the set of **worlds** that correspond to the interpretations of L. We call Ω the **frame of discernment** (the frame for short). **Propositions** identify the subsets of Ω. Let $\top$ be the tautology and $\bot$ be the contradiction. For any proposition X, let $[\![X]\!] \subseteq \Omega$ be the set of worlds identified by X. Let A be a subset of Ω, then f_A is any proposition that identifies A. So $A=[\![f_A]\!]$, $\emptyset=[\![\bot]\!]$ and $\Omega=[\![\top]\!]$. The domain of Cr are sets of worlds in Ω. By definition the actual world ϖ is an element of Ω. $\forall A \subseteq \Omega$, Cr(A) quantifies Y's beliefs at time t that $\varpi \in A$.

In L, two propositions A and B are **logically equivalent**, denoted A≡B, iff $[\![A]\!]=[\![B]\!]$. Beside the logical properties, there is another concept of equivalence related to the evidential corpus EC_t^Y of Y at time t. This property is qualified as doxastic in order to contrast it from its logical counterparts. Let $[\![EC_t^Y]\!]$ represents the set of worlds where all propositions deduced on Ω from EC_t^Y are true. All the worlds in Ω not in $[\![EC_t^Y]\!]$ are accepted as 'impossible' for Y at time t. Two propositions A and B are said to be **doxastically equivalent** for Y at time t, denoted A≅B, if $[\![EC_t^Y]\!]\cap[\![A]\!] = [\![EC_t^Y]\!]\cap[\![B]\!]$

For A⊆Ω, $\overline{A}$ denotes the set of worlds in $[\![EC_t^Y]\!]$ not in A, hence $\overline{A}=[\![EC_t^Y]\!]\cap[\![\neg f_A]\!]$.

Let Π be a partition of Ω. Given the elements of the partition Π, we build ℜ, the **Boolean algebra** of the subsets of Ω based on Π. Each set of worlds in Ω that is an elements of the partition Π on which the algebra ℜ is based is called an **atom** of ℜ. Given ℜ, the number of atoms in a set A∈ℜ is the number of atoms of ℜ that are included in A. We call the pair (Ω, ℜ) a **propositional space**.

3. The credibility function.

Let Y be an agent. Let EC_t^Y be Y's evidential corpus at time t. Let Ω be the frame of discernment on which Y entertains his beliefs concerning the answer ϖ to a question of interest, i.e. Y allocates his beliefs at time t to the elements of ℜ, an algebra defined on Ω. It is postulated that the beliefs held by Y are quantified by a point-valued "credibility" function Cr which maps ℜ into [0, 1], is uniquely defined by (EC_t^Y,Y, t), is monotonic for inclusion, reaches its lower limit for Ø and its upper limit for Ω. The triple (Ω, ℜ, Cr) is called a **credibility space**. The index in $(\Omega, \Re, Cr)_{EC_t^Y}$ denotes the evidential corpus on which Cr is based.

The first axiom assumes that propositions that are doxasticaly equivalent for Y at time t receive the same beliefs (Kyburg, 1987a).

Axiom A1: Equi-credibility of doxasticaly equivalent propositions.
Suppose two credibility spaces $(\Omega, \Re_i, Cr_i)$, i=1,2 induced by EC_t^Y. Let $A_1\in\Re_1$, $A_2\in\Re_2$ Let f_{A1} and f_{A2} be any proposition that identifies A_1 and A_2. Let $f_{A1}\cong f_{A2}$. Then $Cr_1(A_1) = Cr_2(A_2)$.

Next, in Smets (1990b) we prove that the set of credibility functions defined on a propositional space (Ω, ℜ) is a convex set, i.e., if Cr_1 and Cr_2 are two credibility functions defined on ℜ, then $\alpha.Cr_1 + (1-\alpha).Cr_2$, $\alpha\in[0,1]$, is also a credibility function on ℜ. We also derive the pignistic transformation, i.e., the transformation that permits the construction of the probability

function needed for decision making. We prove that probability functions are credibility functions (Smets 1992b).

4. Coarsening.

We study the impact that would result on Y's beliefs from a change of the algebra on which Cr is initially defined. These changes of algebra are said '**uninformative**' in that they do not induce a change in the evidential corpus EC_t^Y on which Y's beliefs at t were based. Two types of change are considered: the coarsening and the refinement (see section 5). Intuitively the first corresponds to a grouping together of the atoms of $\Re$ whereas the second corresponds to a splitting of the atoms of $\Re$.

Coarsening			Refinement		
$\Re$	C	$\Re'$	$\Re$	R	$\Re'$
ω_1					z_1
ω_2		z_1	ω_1		z_2
ω_3			ω_2		z_3
ω_4		z_2			z_4
ω_5			ω_3		z_5
ω_6		z_3			z_6

Figure 1: Examples of coarsening C and refinement R from $\Re$ to $\Re'$ which atoms are respectively the ω_i's and the z_i's.

Let $(\Omega, \Re, Cr)$ be the credibility space induced by EC_t^Y. Let C be a mapping from $\Re$ to $\Re'$, an algebra defined on the same frame Ω, such that one to several atoms of $\Re$ are mapped into one atom of $\Re'$ and each atom of $\Re$ is mapped into one and only one atom of $\Re'$. Let $C(\omega)$ be the atom of $\Re'$ on which the atom ω of $\Re$ is mapped, and $\forall A \in \Re$, $C(A) = (\cup C(\omega) : \omega \in A)$. The mapping C is called a **coarsening**. For $A' \in \Re'$, $C^{-1}(A')$ is the union of the atoms of $\Re$ which are mapped by C onto an atom of A'. Let Cr' be the credibility function induced on $\Re'$ from EC_t^Y. The same evidential corpus EC_t^Y induces both Cr and Cr', the only difference being in the granularity of the algebras. Hence Cr and Cr' are strongly related. That relation between Cr and Cr' is immediate thanks to axiom A1.

Theorem 1. Let the credibility space $(\Omega, \Re', Cr')_{EC_t^Y}$ be derived from $(\Omega, \Re, Cr)_{EC_t^Y}$ through the uninformative coarsening C. Then for $A' \in \Re'$:

$$Cr'(A') = Cr(C^{-1}(A')).$$

Thanks to theorem 1, the granularity of Ω in $(\Omega, \Re, Cr)$ becomes essentially irrelevant. The only constraint induced by $\Re$ on Ω is that each element of Ω is included in one atom of $\Re$ and

each atom of $\Re$ contains at least one element of Ω. So one has full freedom to change Ω provided Ω is compatible with $\Re$ (i.e. satisfy the constraints induced by $\Re$).

5 . Refinement.

Let $(\Omega, \Re)$ be a propositional space. Suppose a mapping R from $(\Omega, \Re)$ to another propositional space $(\Omega', \Re')$ where $\Re'$ is an algebra such that each atom of $\Re$ is mapped into one or several atoms of $\Re'$ and each atom of $\Re'$ is derived from one and only one atom of $\Re$. The structure of the frames of discernment Ω and Ω' is not important for our presentation provided they are compatible with $\Re'$. Therefore Ω and Ω' can both be re-defined such that they are equal (and denoted Ω^*) and their elements correspond to the atoms of $\Re'$. Let R(A) be the image of $A \in \Re$ in $\Re'$, and let $R(\emptyset)=\emptyset$. The mapping R is called a **refinement**.

Let $(\Omega, \Re, Cr)_{EC_t^Y}$ be a credibility space based on EC_t^Y and let $(\Omega^*, \Re')$ be the propositional space derived from $(\Omega, \Re)$ by the uninformative refinement R. Adapt Ω of the first credibility space into Ω^*, so both credibility spaces (the ones before and after refinement) share the same Ω^*. We must determine what is the credibility function Cr' induced on $\Re'$ by EC_t^Y, i.e. by Cr. As $\forall A \in \Re$, $f_A \cong f_{R(A)}$, then Cr'(R(A)) = Cr(A) by axiom A1. But Cr' is not defined on the elements of $\Re'$ that are not the image of some elements of $\Re$ under R. Thus we must define the credibility Cr' on $\Re'$ for these elements of $\Re'$ given the credibility Cr on $\Re$ and the uninformative refinement R.

In order to explain the construction of Cr', consider the following illustrative example. Let ϖ be Paul's age, $\Omega = [0, \infty)$, and $\omega_1 = [0,20)$, $\omega_2 = [20, 40)$, $\omega_3 = [40, \infty)$ be the three atoms of $\Re$. Let Cr quantifies Y's beliefs on $\Re$ based on EC_t^Y. Let P_1, P_2 and P_3 be three propositions. Y does not know what are these three propositions, he only knows that one and only one of them is true. Let a refinement R from $\Re$ to $\Re'$ with $R(\omega_1) = \omega_1$, $R(\omega_2) = \omega_2$, and $R(\omega_3) = \{X_1, X_2, X_3\}$ where $X_i = '\omega_3$ and P_i'.

The problem is to determine Cr' given the evidential corpus EC_t^Y that leads to the construction of Cr on $\Re$ and the knowledge of the refinement R. Consider the value of $Cr'(\omega_1 \cup X_i)$. The uninformativeness of R is tanslated into the requirement:

$$Cr'(\omega_1 \cup X_1) = Cr'(\omega_1 \cup X_2) = Cr'(\omega_1 \cup X_3).$$

Suppose now that one uses another uninformative refinement R" from $(\Omega^*, \Re)$ to $(\Omega^*, \Re")$ such that $R"(\omega_1) = \omega_1$, $R"(\omega_2) = \omega_2$, and $R"(\omega_3) = \{Y_1, Y_2\}$ where $f_{Y_1} \cong f_{X_1}$, $f_{Y_2} \cong f_{X_2 \cup X_3}$. The credibility Cr" over $\Re"$ is such that:

$$Cr"(\omega_1 \cup Y_1) = Cr"(\omega_1 \cup Y_2).$$

By axiom A1:

$$Cr"(\omega_1 \cup Y_1) = Cr'(\omega_1 \cup X_1).$$

$Cr''(\omega_1 \cup Y_2) = Cr'(\omega_1 \cup X_2 \cup X_3)$.

Therefore one obtains:
$Cr'(\omega_1 \cup X_2) = Cr'(\omega_1 \cup X_1) = Cr''(\omega_1 \cup Y_1) = Cr''(\omega_1 \cup Y_2) = Cr'(\omega_1 \cup X_2 \cup X_3)$
So $Cr'(\omega_1 \cup X)$ is equal for all $X \subset R(\omega_3)$ (where $X \subset Y$ means that $X \subseteq Y$ but $X \neq Y$). The impact of an uninformative refinement is formalized by axiom R1 where we must only postulate the equalities on the atoms, as the other equalities are deduced from axiom A1 and the same argument as the one just presented.

Axiom R1: Let $(\Omega, \Re, Cr)_{EC_t^Y}$ be a credibility space and let R be a refinement from $(\Omega, \Re)$ to $(\Omega', \Re')$. Let ω be a given atom of $\Re$ and $B \in \Re'$ where $B \cap R(\omega) = \emptyset$. Let X_i: i=1, 2...n be the atoms of $\Re'$ included in $R(\omega)$. Let $(\Omega', \Re', Cr)_{EC_t^Y}$ be the credibility space induced from $(\Omega, \Re, Cr)_{EC_t^Y}$ by R on $(\Omega', \Re')$. Then

$$Cr'(B \cup X_i) = Cr'(B \cup X_j) \qquad \forall i,j \in \{1,\ 2...n\}.$$

The next theorem just formalize the intuitive proof detailled before introducing axiom R1.

Theorem 2: Under axiom R1 conditions, for any $\omega \in \Omega$, any $B \in \Re'$ and with $B \cap R(\omega) = \emptyset$, $Cr'(B \cup X)$ is constant for all $X \subset R(\omega)$.

The property represents a serious departure from what is encountered in classical probability theory. That departure reflects the difficulty for probability theory to deal with states of total ignorance as those encountered with the P_i's. In fact, in probability theory, one cannot accept simultaneously axiom R1 for any uninformative refinement R and axiom A1. In probability theory, the introduciton of a uninformative refinement is accompanied by the introduction in EC_t^Y of the information that tells for each atoms ω of $\Re$ how the probability given to ω is distributed among the new atoms of $R(\omega)$. Often an equidistribution of the probability given to ω among the atoms of $R(\omega)$ is assumed, but any distribution is acceptable. The information about that distribution is linked to the refinement, therefore EC_t^Y is updated, and thus the refinement is not uninformative as required. The uninformative refinements we consider in axiom R1 are those that correspond only to a change in the granularity of the algebra on which our credibility function is build, without changing the evidential corpus EC_t^Y on which Y's beliefs are based.

Another axiom must be assumed to determine Cr' after uninformative refinement. Let $A \in \Re$ and let n distinct atoms ω_i, i=1, 2...n of $\Re$, none being included in A. Let $X_i \subset R(\omega_i)$, i=1, 2...n. We postulate that $Cr'(R(A) \cup X_1 \cup X_2 ... \cup X_n)$ depends only on the beliefs $Cr(A \cup B)$ given to the union of A with each subset B of $\omega_1 \cup \omega_2 \cup .. \cup \omega_n$.

Axiom R2: Let $(\Omega, \Re, Cr)_{EC_t^Y}$ be the credibility space based on EC_t^Y. Let R be a uninformative refinement from $(\Omega, \Re)$ to $(\Omega', \Re')$. Let $(\Omega', \Re', Cr')_{EC_t^Y}$ be the credibility

space induced from $(\Omega, \Re, Cr)_{EC_t^Y}$ by R on $(\Omega', \Re')$. Let $A \in \Re$. Let ω_i: i=1,2...n, be n different atoms of $\Re$ with $A \cap \omega_i = \emptyset$. Let X_i be any element of $\Re'$ strictly included in $R(\omega_i)$: i=1,2...n. Then there is a g function such that:

$$Cr'(R(A) \cup X_1 \cup X_2 ... \cup X_n) = g(\{Cr(A \cup B): B \in \Re, B \subseteq \omega_1 \cup \omega_2 \cup .. \cup \omega_n\})$$

In fact, axiom R2 could be simplified by requiring only that $Cr'(R(A) \cup X_1 \cup X_2 ... \cup X_n)$ does not depend on Cr(X) when $X \cap (A \cup \omega_1 \cup \omega_2 \cup .. \cup \omega_n) = \emptyset$. Intuitively that property is of the same nature as the one underlying Axiom A1: irrelevant credibilities should not interfere with the other credibilities. Given that requirement, axiom A1 permits to deduce axiom R2. The gain is not worth the needed proof.

From axioms R1 and R2, it is possible to prove that Cr' must satisfy one of the following three relations:

$$Cr'(R(A) \cup X_1 \cup X_2 ... \cup X_n) = Cr(A)$$

$$Cr'(R(A) \cup X_1 \cup X_2 ... \cup X_n) = Cr(A \cup \omega_1 \cup \omega_2 \cup .. \cup \omega_n)$$

$$Cr'(R(A) \cup X_1 \cup X_2 ... \cup X_n) = \max(Cr(A), Cr(\omega_1), Cr(\omega_2), ...Cr(\omega_n))$$

The first solution is the one encountered when Cr is a belief function, and the second is the one encountered when Cr is a plausibilty function. The third solution will not satisfy the conditioning axioms (but find an application in possibility theory).

6. Updating.

Let EC_t^Y be the evidential corpus held by Y at time t and let $(\Omega, \Re, Cr)_{EC_t^Y}$ be the credibility space characterizing Y's beliefs at time t about which subsets of worlds of Ω among those in $\Re$ include the actual world ϖ. Suppose Y expends EC_t^Y by adding the evidence Ev_A compatible with EC_t^Y that implies that all worlds in $\overline{A} \subseteq \Omega$ are impossible, i.e. that the actual world ϖ is not in $\overline{A}$, or equivalently that $f_A \cong \top$. How does Y update his beliefs given the addition of Ev_A to EC_t^Y? Let Cr_A be the conditional credibility function that results from the addition of Ev_A to EC_t^Y. It is postulated in axiom M1 that Cr_A is derived from the credibility function Cr based on EC_t^Y.

Axiom M1: Let $(\Omega, \Re\ Cr)_{EC_t^Y}$ be a credibility space based on EC_t^Y. Let Ev_A be an evidence compatible with EC_t^Y that implies that $f_A \cong \top$. Let $(\Omega, \Re, Cr_A)_{EC_t^Y \cup \{Ev_A\}}$ be the credibility space based on $EC_t^Y \cup \{Ev_A\}$. Then Cr_A depends only on Cr and A.

To derive the conditioning process, we will use the idea of iterated conditioning. For $A,B \subseteq \Omega$, let f_A, f_B and $f_{A \cap B}$ be the propositions that denotes the sets of worlds A, B and $A \cap B$. Suppose you learn 1) that $f_A \cong \top$ and then that $f_B \cong \top$, or 2) that $f_B \cong \top$ and then that $f_A \cong \top$, or 3) directly that $f_{A \cap B} \cong \top$. The final conditional belief should be the same in these three cases. This

requirerment introduces enough constraints to derive the mathematical structure of the updating process.

We prove that the conditional credibility function $Cr_A(B)$ depends only on some of the elements of $\Re$.

Theorem 3: Let $(\Omega, \Re\ Cr)_{EC_t^Y}$ be a credibility space based on EC_t^Y. For $A \in \Re$, let Ev_A be an evidence compatible with EC_t^Y that implies that $f_A \cong \mathsf{T}$. Let $(\Omega, \Re, Cr_A)_{EC_t^Y \cup \{Ev_A\}}$ be the credibility space based on $EC_t^Y \cup \{Ev_A\}$. Then there is a f function such that Cr_A satisfies:
1: $Cr_A(B) = 0 \qquad \forall B \subseteq \overline{A},\ B \in \Re$
and $\forall B \in \Re$
2: $Cr_A(B) = Cr_A(B \cap A)$
3: $Cr_A(B) = f\ (Cr(B \cap A), Cr(\overline{B} \cap A), Cr(\overline{A}), Cr(A), Cr((B \cap A) \cup \overline{A}), Cr((\overline{B} \cap A) \cup \overline{A}), Cr(\Omega))$

The following theorem formalizes the idea that refining one atom ω of $\Re$ into two new atoms ω_1 and ω_2 in $\Re'$ and conditioning then on $\overline{\omega}_2$ will leave the credibility function unchanged (except the algebra has changed). To illustrate the underlying idea, consider the following example dealing with Paul's age ϖ. Consider Y's beliefs about ϖ, im particular $Cr_{\varpi<40}(\varpi<20)$. Then consider a refinement R from $(\Omega, \Re)$ to $(\Omega', \Re')$ with $R([0,20)) = [0,20)$, $R([20,40)) = \{([20,40),Q), ([20,40),\neg Q)\}$, $R([40,\infty)) = [40,\infty)$, where Q is a proposition unknown to Y (like the P_i's propositions of section 5). Y builds his credibility function Cr' on $\Re'$. Than Y learns that Q is true. What is Y's beliefs about $\varpi<20$ given $((\varpi<20)$ or $(20 \le \varpi<40$ and $Q))$. We feel it has to be equal to Y's previous belief $Cr_{\varpi<40}(\varpi<20)$ about $\varpi<20$ given $(\varpi<20$ or $20 \le \varpi<40)$. Indeed the Q story becomes irrelevant to Y's beliefs on ϖ, and this is what theorem 4 confirms. Formally one has:

Theorem 4: Let $(\Omega, \Re, Cr)_{EC_t^Y}$ be the credibility space based on EC_t^Y. Let a uninformative refinement R from $(\Omega, \Re)$ to $(\Omega', \Re')$ such that each atom of $\Re$ is mapped onto itself in $\Re'$, except one atom ω of $\Re$ that is refined into ω_1 and ω_2 by R. Let Cr' be the credibility function derived from Cr on $\Re'$ by R. Suppose the conditioning of Cr' on $\overline{\omega}_2$, then
$\forall A \in \Re \qquad Cr'_{\overline{\omega}_2}(R(A) \cap \overline{\omega}_2) = Cr(A)$

We introduce two other axioms. The first (M2) eliminates degenerated solutions. The second (M3) says that if fotr $X,Y \in \Re$, $X,Y \subseteq A \in \Re$, $X \cap Y = \emptyset$, $Cr(X \cup Y) = Cr(X) + Cr(Y)$ then the conditional crfedibiulkty function Cr_A obeys $Cr_A(X \cup Y) = Cr_A(X) + Cr_A(Y)$ (just as in probability theory). We do not require that normalization is preserved, as it will correspond to a particular case of our conditioning operator. The axioms M1, M2 and M3 combined with the axioms about refinement are sufficient to derive the explicit structure of the conditional credibility functions.

Axiom M2: Non-degenerated solutions. $Cr_A(B)$ is not constant for all $A \in \Re$.

Axiom M3: Additivity preservation. Let $(\Omega, \Re, Cr)$ be a credibility space. If the credibility function Cr is additive, then additivity is preserved after conditioning.

Theorem 5. Let $(\Omega, \Re, Cr)_{EC_t^Y}$ be the credibility space based on EC_t^Y. Let $\{\omega_1, \omega_2, ...\omega_n\}$ be the set of atoms of $\Re$. Let a uninformative refinement R from $(\Omega, \Re)$ to $(\Omega', \Re')$. Let $(\Omega', \Re', Cr')_{EC_t^Y}$ be the credibility space derived from $(\Omega, \Re, Cr)_{EC_t^Y}$ by R. Let $I \subseteq \{1,2,\ldots n\}$. For $i \in I$, let $X_i \subset R(\omega_i)$. Let $A \in \Re$ and $\omega_i \cap A = \emptyset$, $\forall i \in I$. The refinement and the conditioning process admit only two solutions:
The minimal solution is:

$$Cr_A(B) = \frac{Cr(B \cup \overline{A}) - Cr(\overline{A})}{Cr(\Omega) - Cr(\overline{A})} Cr_A(A) \qquad Cr'(R(A) \cup (\underset{i \in I}{\cup} X_i)) = Cr(A)$$

The maximal solution is:

$$Cr_A(B) = \frac{Cr(A \cap B)}{Cr(A)} Cr_A(A) \qquad Cr'(R(A) \cup (\underset{i \in I}{\cup} X_i)) = Cr(A \cup (\underset{i \in I}{\cup} \omega_i))$$

The qualification of the solutions as minimal and maximal results from the fact they correspond to the extremal solutions among the possible solutions. Indeed the following inequalities are required by the monotonicity for inclusions and axiom A1:

$$Cr(A) \leq Cr'(R(A) \cup (\underset{i \in I}{\cup} X_i)) \leq Cr(A \cup (\underset{i \in I}{\cup} \omega_i)).$$

The minimal and the maximal solutions of theorem 5 are in fact dual. We define the co-credibility function CoCr on $\Re$ induced by a credibility function Cr on $\Re$ by:

$$CoCr(A) = Cr(\Omega) - Cr(\overline{A}) \qquad \forall A \in \Re$$

If the credibility function satisfies the minimal solution, its related cocredibility function satisfies the maximal solution, and vice versa. Therefore there is in practice only one credibility function, as the other solution is always its dual. That duality relation is the one encountered between the belief functions and the plausibilty functions.

The solutions for $Cr_A(B)$ depend on the value of $Cr_A(A)$. Three particular cases merit consideration.

$Cr_A(A) = 1$: the solutions are those obtained by the normalized Dempster's rule of conditioning, i.e. the solutions described in the transferable belief model under closed-world assumption.

$Cr_A(A) = Cr(\Omega) - Cr(\overline{A})$ in the minimal solution and $Cr_A(A) = Cr(A)$ in the maximal solution: the solutions are those obtained by he unnormalized Dempster's rule of conditioning as described in the transferable belief model under open-world assumption.

$Cr_A(A) = Cr(\Omega)$: the solutions corresponds to those obtained by a partially renormalized Dempster's rule of conditioning. It fits with the idea that the belief initially given to Ω is preserved by proportionaly reallocating the belief given initially to $\overline{A}$. The nature of such a solution is nevertheless not very clear to us except if $Cr(\Omega)=1$ in which case it is equivalent to the first solution.

If initially the belief is quantified by a probability function, and if one accepts $Cr_A(A) = 1$, then both the maximal and the minimal solutions of theorem 5 are identical and correspond to the conditioning rule encountered in probability theory.

Gärdenfors (1988) proposed two compelling properties for probabilistic revision functions, that are not simultaneously satisfiable in probability theory. In the context of the minimal solution, they translate into:

Homomorphisme:
If $Cr(\overline{A})<Cr(\Omega)$ and $Cr = p.Cr' + (1-p).Cr''$, $p\in[0,1]$, then $Cr_A = p.Cr'_A + (1-p).Cr''_A$.

Preservation:
If $Cr(\overline{A})<Cr(\Omega)$ and $Cr(B) = Cr(\Omega)$, then $Cr_A(B) = Cr_A(A)$.

Homomorphisme is satisfied only by the unnormalized Dempster's rule of conditioning. It fails whenever a normalization factor is introduced through a division.

Preservation is satisfied by both solution for conditioning. One could have considered that $Cr_A(B) = Cr(\Omega)$ would be required by the preservation. But in order to satisfy both homomorphisme and preservation, one would have to require that: $Cr_A(B) = Cr_A(A) = Cr(\Omega) - Cr(\overline{A})$. These equlities are satisfied in the transferable belief model for quantified beliefs where conditioning is performed by the unnormalized Dempster's rule of conditioning (Smets and Kennes, 1990). In that case, the prerequisite $Cr(\overline{A})<Cr(\Omega)$ can even be relaxed, but if $Cr(\overline{A})=Cr(\Omega)$, then $Cr_A(B)=0\ \forall B\in\Re$, so even $Cr_A(\Omega)=0$, a belief that describes a state of complete contradiction not dissimilar to the one encountered in logic when one simultaneously knows something and its contrary. This problem is studied in Smets (1992).

7. Credibility functions and belief functions.

The aim of this paper was to determine which properties are required in order to justify that beliefs should be quantified by belief functions. Up to here the credibility functions are not

restricted to being belief functions. To achieve our aims, we introduce a closure property that reduces the credibility functions to belief functions.

Let a propositional space (Ω, $\Re$). The only relevant information in $\Re$ to define the set of all credibility functions defined on $\Re$ is the number r of atoms in $\Re$ (r = $|\Re|$) so we write C_r for the set of credibility functions defined on an algebra with r atoms and that satisfy to the minimal solutions of theorem 5. Let $\mathbf{I}_\Omega$ be the 'vacuous' credibility function on (Ω, $\Re$) where $\mathbf{I}_\Omega(X)=0$ $\forall X \subseteq \Omega$, $X \neq \Omega$, and $\mathbf{I}_\Omega(\Omega)=1$.

Let B_r be the set of belief functions defined on an algebra with r atoms. B_r is closed under conditioning. Furthermore all elements of B_r can be generated from the vacuous credibility function $\mathbf{I}_\Omega \in C_r$ by appropriate conditioning and simplex combinations (a simplex combination is a convex combination except that the sum of the weights may be in [0, 1] instead of being 1).

Let R_r be the set of refinements R_i , i=1, 2..., from (Ω, $\Re$) to (Ω', $\Re'$) where $|\Re|$ = r and $|\Re'|$ = r+1, i.e. one and only one atom of $\Re$ has been refined into two atoms of $\Re'$. For $Cr \in C_r$, $R_i \in R_r$, let $R_i(Cr) \in C_{r+1}$ be the credibility function defined on $\Re'$ after aplying the refinement R_i by using the minimal solution given in theorem 5.

Let Ext(C_r) be the set of credibility functions on $\Re'$ that can be obtained by convex combinations of the credibility functions generated on $\Re'$ by the application of the refinement operators in R_r on the credibility functions in C_r. Let D(Ext(C_r)) be the closure of Ext(C_r) that contains all the credibility functions that can be obatined from those in Ext(C_r) through conditioning (by the minimal solution of theorem 5) and simplex combinations. Formally:

$$\text{Ext}(C_r) = \{Cr: Cr \in C_{r+1},\ Cr = \sum_i \alpha_i R_i(Cr_i),\ Cr_i \in C_r,\ R_i \in R_r,\ \alpha_i \geq 0,\ \sum_i \alpha_i = 1\}$$

$$D(\text{Ext}(C_r)) = \{Cr: Cr \in C_{r+1},\ Cr = \sum_i \alpha_i Cr_{iA_i},\ \alpha_i \geq 0,\ \sum_i \alpha_i = 1,$$
$$Cr_{iA_i} \text{ is the (minimal) conditioning on } A_i \in \Re' \text{ of } Cr_i \in D(\text{Ext}(C_r))\}$$

The problem is to decide if $C_{r+1} = D(\text{Ext}(C_r))$ or not? We cannot prove it, but we feel it can be postulated. We feel reasonable to assume that any credibility function in C_{r+1} could be derived from some credibility functions in C_r through refinement, conditioning and simplex combinations.

The Closure Axiom : $C_{r+1} = D(\text{Ext}(C_r))$

This axiom has the immediate consequence that $B_r = C_r$, i.e. every credibility function in C_r is a belief function.

Theorem 6.
1: $B_r = C_r$.
2: credibility functions that satisfy the minimal solution for conditioning and refinement are belief functions.
3: credibility functions that satisfy the maximal solution for conditioning and refinement are plausibilty functions.

8. Conclusions.

We have shown under which conditions beliefs are quantified by belief functions at the credal level, i.e. where beliefs are entertained. These conditions seem acceptable, and therefore they provide a justification for the transferable belief model to quantify some one's beliefs (Smets and Kennes, 1990).

One might be tempted to consider some of the axioms as unreasonable. It happens most if not all that the axioms are satisfied in probability theory (except for the simultaneous satisfaction of the homomorphism and the preservation properties). Therefore the rejection of our axioms might lead to a simultaneous rejection of probability theory! In fact probability functions are special cases of normalized belief functions.

The nature and use of the transferable belief model is detailed in Smets and Kennes (1990). In Smets (1990b) we show and explain what is the probability function that must be used to make decisions given the beliefs entertained at the credal level. In Smets (1990a) we show what is the justification of the Dempster's rule of combination (see also Klawonn and Schwecke, 1992, Klawonn and Smets 1992). The concept of distinctness is described in Smets (1992c). The meaning of $Cr(\Omega)<1$ is analysed in Smets (1992a). The combination of the belief functions induced by two non-distinct pieces of evidence are already tackled in Kennes (1991) and Smets (1986).

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